Linear spaces of symmetric matrices and combinatorial algebraic geometry

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LSSM study group

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Two questions about LSSMs

Let $\mathcal{L} \subset \mathbb{C}^{n\times n}$ be a d-dimensional linear space of symmetric matrices.

Question A

What is the degree of the variety $\mathcal{L}^{-1}$ obtained by inverting all matrices in $\mathcal{L}$?

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ML-degrees

ML-degree of the linear concentration model

The *ML-degree* of an LSSM $\mathcal{L} \subset \mathrm{Sym}_2 \mathbb{C}^n$ is the number of critical points of the function $K \mapsto \log(\det K) - \mathrm{tr}(S \cdot K)$ for S generic.

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Exercise

The critical points are precisely the matrices $K \in \mathcal{L}$ for which $\langle K^{-1}, X \rangle = \langle S, X \rangle$ for all $X \in \mathcal{L}$. (Here $\langle X, Y \rangle := \mathrm{tr}(X \cdot Y)$).

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Hint

Write $\mathcal{L} = \mathrm{Span}(K_1, \ldots, K_d)$, so that $K = \sum_i \lambda_i K_i$, and put all partial derivatives $\frac{\partial}{\partial \lambda_j}$ equal to 0. Use Jacobi's formula: `https://en.wikipedia.org/wiki/Jacobi's_formula`.

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The ML-degree of an LSSM $\mathcal{L} \subset \mathrm{Sym}_2 \mathbb{C}^n$ is the number of pairs (K, Σ) with

$$\Sigma \cdot K = Id_n,\ K \in \mathcal{L},\ \Sigma - S \in \mathcal{L}^{\perp},$$

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The ML-degree of a general d-dimenional LSSM $\mathcal{L} \subset \mathrm{Sym}_2 \mathbb{C}^n$ is the number of pairs (K, Σ) with

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where $S \in \mathrm{Sym}_2 \mathbb{C}^n$ generic and $\mathcal{M}$ is a general LSSM of codimension d.

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For special $\mathcal{L}$ (e.g. Gaussian graphical models), there are 2 interesting numbers to compute: $\deg(\mathcal{L}^{-1})$ and the ML-degree.

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Question A, projectivized

$\phi(n,d)$ is the number of pairs $(K,\Sigma) \in \mathbb{P}(S^2V) \times \mathbb{P}(S^2V^*)$ with

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$X \subset \mathbb{P}(S^2V) \times \mathbb{P}(S^2V^*)$ the variety parametrized by (K, K^{-1}). The points of X are all pairs $(K,\Sigma) \in \mathbb{P}(S^2V) \times \mathbb{P}(S^2V^*)$ with $K \cdot \Sigma = Id_n$ or $K \cdot \Sigma = 0$.

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The algebraic degree of SDP

The KKT equations

Let $\mathcal{V} \subseteq \mathcal{U} \subseteq \mathrm{Sym}_2(\mathbb{R}^n)$ be LSSM's, with $\dim(\mathcal{V}) = d - 1$ and $\dim(\mathcal{U}) = d + 1$. The KKT equations can be written as:

$$X \in \mathcal{V}^\perp,\ Y \in \mathcal{U},\ X \cdot Y = 0.$$

For generic $\mathcal{U}$ and $\mathcal{V}$, the number of solutions (X, Y) with $\mathrm{rk}(X) = r$ and $\mathrm{rk}(Y) = n - r$ is known as the *algebraic degree of semidefinite programming*, written $\delta(d, n, r)$.

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$\delta(d, n, r)$ is the number of solutions to:

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Fact

Let $\mathcal{L}_r \subset \mathcal{L}$ be the variety of all rank $\leq r$ matrices in $\mathcal{L}$, and let $\mathcal{L}_r^*$ be the dual variety. If $\mathcal{L}_r^*$ a hypersurface, then it's degree is equal to $\delta(d, n, r)$.

Key point: for $Y \in \mathcal{D}_r \subseteq \mathbb{P}(S^2V)$, the hyperplanes tangent to $\mathcal{D}_r$ are precisely the $X \in \mathbb{P}(S^2V^*)$ with $X \cdot Y = 0$.

The KKT equations

Let $V = \mathbb{C}^n$ and let $\mathcal{U} \subseteq \mathbb{P}(\mathrm{Sym}_2(V))$, $\mathcal{W} \subseteq \mathbb{P}(\mathrm{Sym}_2(V^*))$ be general LSSM's, with $\mathrm{codim}(\mathcal{V}) = d - 1$ and $\dim(\mathcal{U}) = d$.
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Complete quadrics

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Definition

The space $\Phi(V)$ of complete quadrics is the closure of the image of the set of invertible matrices under the map

$$\varphi : \mathbb{P}(S^2V) \hookrightarrow \mathbb{P}\left(S^2V\right) \times \mathbb{P}\left(S^2(\bigwedge^2 V)\right) \times \cdots \times \mathbb{P}\left(S^2(\bigwedge^{n-1} V)\right),$$

sending a matrix A to $(A, \bigwedge^2 A, \dots, \bigwedge^{n-1} A)$.

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$\Phi(V)$ is a smooth variety such that

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$$
\begin{aligned}
S_r &= \{(A_1,\dots,A_{n-1}) \in \Phi(V) \mid \operatorname{rk} A_r = 1\} \\
&= \operatorname{cl}\{(A_1,\dots,A_{n-1}) \in \Phi(V) \mid \operatorname{rk} A_1 = r\} \\
&= \operatorname{cl}\{(A_1,\dots,A_{n-1}) \in \Phi(V) \mid \operatorname{rk} A_{n-1} = n-r\}.
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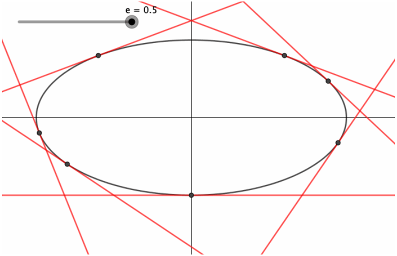
in the Plücker coordinates of a $(k-1)$-plane.
- So a point in $\mathcal{A} = (A_1, \ldots, A_{n-1}) \in \Phi(V)$ is given by a collection of (possibly nonsmooth) quadrics $Q_i = Q(A_{i+1}) \subset \mathbb{G}(i, \mathbb{P}V^*)$.
- For a general $\mathcal{A}$, all Q_i are smooth, and Q_i is the space of i-planes tangent to Q_0.

Example: n=2

$$\left(\begin{pmatrix} -\varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}\right)$$



is a point in $\Phi(V)$:

- Q_0: $\varepsilon x_0^2 = \varepsilon x_1^2 + x_2^2$,
  a smooth conic in $\mathbb{P}(V^*)$
- Q_1: $\beta_0^2 = \beta_1^2 + \varepsilon\beta_2^2$,
  a smooth conic in $\mathbb{P}(V)$

(i.e. a line $\beta_0 x_0 + \beta_1 x_1 + \beta_2 x_2 = 0$ is tangent to Q_0 if and only if $\beta_0^2 = \beta_1^2 + \varepsilon\beta_2^2$.)

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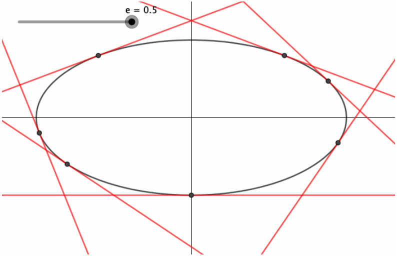

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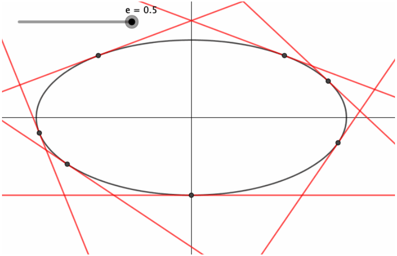


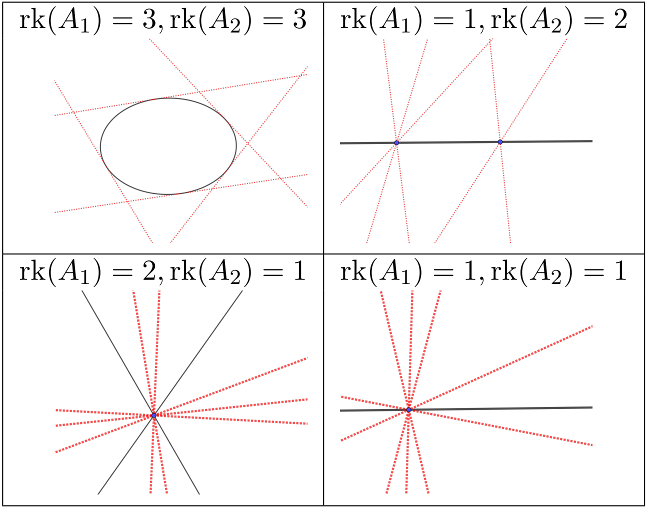
is a point in $\Phi(V)$:

- Q_0: $x_2^2 = 0$,
  a double line in $\mathbb{P}(V^*)$
- Q_1: $\beta_0^2 = \beta_1^2$,
  two lines in $\mathbb{P}(V)$

(i.e. a line $\beta_0 x_0 + \beta_1 x_1 + \beta_2 x_2 = 0$ is "tangent to Q_0" if and only if $\beta_0 = \pm\beta_1$, if and only if the line goes through $[1 : 1 : 0]$ or $[1 : -1 : 0]$.)

Complete quadrics

For $n = 3$, there are 4 types of complete quadrics $(A_1, A_2) \in \Phi(V) \subset \mathbb{P}(S^2V) \times \mathbb{P}(S^2V^*)$:



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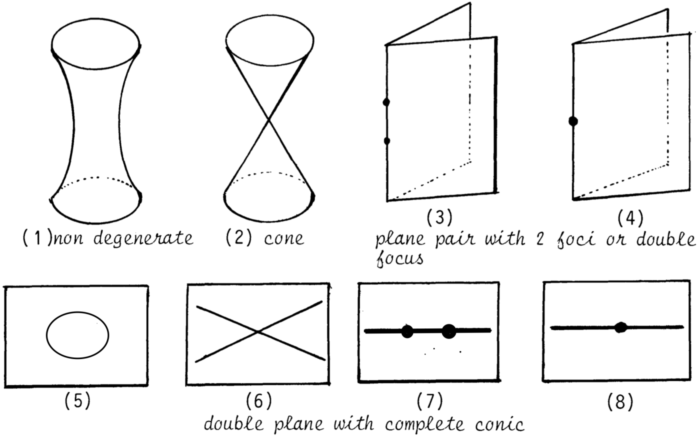

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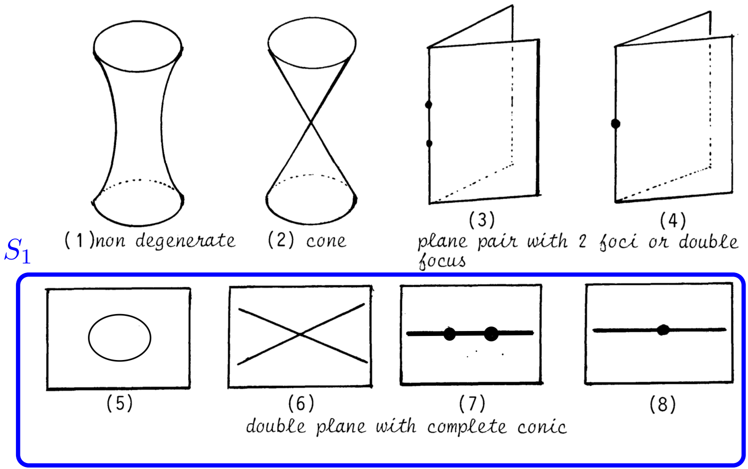

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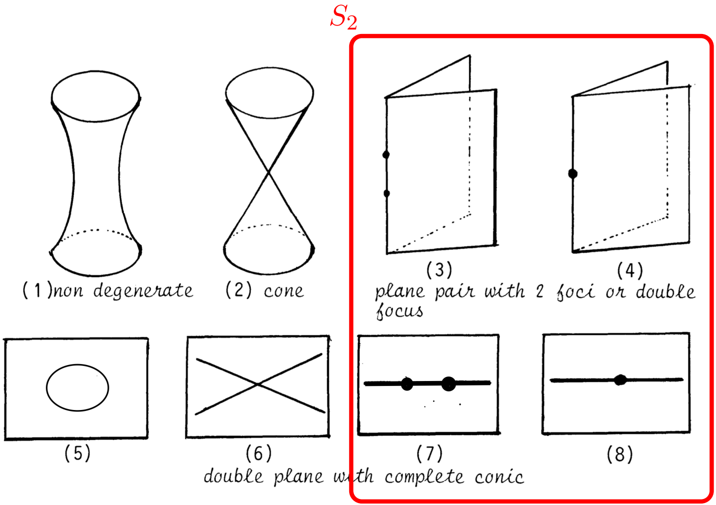

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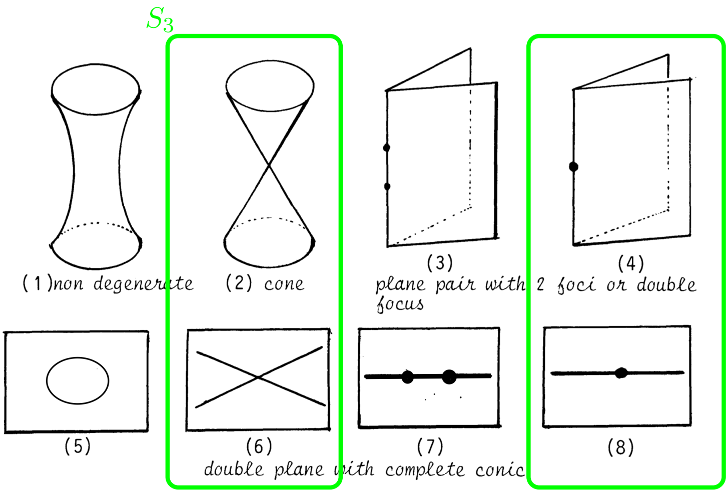

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- $\delta(d,n,r) = |\pi_1^{-1}(\mathcal{U}) \cap \pi_{n-1}^{-1}(\mathcal{V}) \cap S_r|$ is the number of complete quadrics $\mathcal{A} \in S_r$ s.t. Q_0 contains $\binom{n+1}{2} - d - 1$ given points and Q_{n-2} contains $d-1$ given points.

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Remark

It can happen that $\delta(d,n,r) = 0$, namely when *Pataki's inequalities* below are not satisfied, or equivalently, when $\mathcal{U}_r^*$ is not a hypersurface.

$$\binom{n-r+1}{2} \leq d, \binom{r+1}{2} \leq \binom{n+1}{2} - d$$

Enumerative geometry by example

Suppose somebody asks you the following question:

> *"How many lines in $\mathbb{P}^3$ intersect four given lines?"*

Enumerative geometry by example

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- Intersecting subvarieties corresponds to taking products in $A(\mathbb{G}(1,3))$.
- So our problem becomes: find the class $\alpha = [\{\Gamma \mid \Gamma \cap L \neq \emptyset\}] \in A(\mathbb{G}(1,3))$ and compute α^4.

Computing $\phi(n, d)$ and $\delta(d, n.r)$

In the Chow ring $A(\Phi(V))$, we have two interesting classes for every $k = 1, \ldots, n-1$.

- The *degeneration class* $\delta_k = [S_k] = [\{(A_1, \ldots, A_{n-1}) \in \Phi(V) \mid \mathrm{rk}(A_k) = 1\}] \in A^1(\Phi(V))$.
- δ_k is the class of all complete quadrics for which $Q_{k-1} \subseteq \mathbb{G}(k-1, \mathbb{P}V^*)$ is a double hyperplane.

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- μ_k is the class of all complete quadrics for which $Q_{k-1} \subseteq \mathbb{G}(k-1, \mathbb{P}(V^*))$ passes through a given element of $\mathbb{G}(k-1, \mathbb{P}(V^*))$.

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We want to compute

$$\delta(d,n,r) = \mu_1^{\binom{n+1}{2}-d-1} \mu_{n-1}^{d-1} \delta_r \text{ and } \phi(n,d) = \mu_1^{\binom{n+1}{2}-d} \mu_{n-1}^{d-1}.$$

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Two approaches:

- There is an algorithm for computing arbitrary products $\mu_1^{a_1}\cdots\mu_{n-1}^{a_{n-1}}\cdot\delta_1^{b_1}\cdots\delta_{n-1}^{b_{n-1}}$ in $A(\Phi(V))$:

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  - Use the relations $2\mu_k = \mu_{k-1} + \delta_k + \mu_{k+1}$.
- The product $\delta(d,n,r)$ is equal to a product of Segre classes of certain line bundles in $Gr(r,n)$.
  - This allows us to prove that for fixed d, r, $\delta(d,n,r)$ and $\phi(n,d)$ are polynomials in n.

Further directions

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- Equivariant K-theory/cohomology of $\Phi(V)$.
- Nongeneric L (Gaussian graphical models).
- Maximum likelihood degree for linear *covariance* models.

Thank you!