

CLASSIFYING ONE-DIMENSIONAL DISCRETE MODELS WITH MAXIMUM LIKELIHOOD DEGREE ONE

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ABSTRACT. We propose a classification of all one-dimensional discrete statistical models with maximum likelihood degree one based on their rational parametrization. We show how all such models can be constructed from members of a smaller class of ‘fundamental models’ using a finite number of simple operations. We introduce ‘chipsplitting games’, a class of combinatorial games on a grid which we use to represent fundamental models. This combinatorial perspective enables us to show that there are only finitely many fundamental models in the probability simplex Δ_n for $n \leq 4$.

1. INTRODUCTION

A *discrete statistical model* is a subset of the simplex $\Delta_n := \{p \in \mathbb{R}^{n+1} \mid \sum_{\nu} p_{\nu} = 1\}$ of probability distributions on $n + 1$ events for some $n \in \mathbb{N}$. In algebraic statistics, we are interested in models which are *algebraic*, meaning that the model is the intersection of Δ_n and some semialgebraic set in \mathbb{R}^{n+1} . Here, models with *maximum likelihood degree one* are of special interest because for these, the maximum likelihood (ML) estimate is a rational function in the entries of the observed data. For more information on the properties and significance of these models, see [2, 4].

Although the above works explain which form these models may take, a general classification seems out of reach. More specifically, we would like to divide the set of all discrete algebraic models with ML degree one contained in the simplex Δ_n into finitely many easy to understand families. But at the time [2] was published, there was no way to do so even for the simplest case $n = 2$.

In this paper we provide such a classification for $n = 2$, and extend this to $n = 3, 4$ in the case where the models in question are *one-dimensional* as algebraic varieties. Since one-dimensional discrete algebraic models with maximum likelihood degree one are the focus of this paper, we use the word ‘model’ to refer specifically to these.

We start by stratifying the set of models in Δ_n by their *degree* as algebraic curves. We find that for a fixed d , there are essentially finitely many ways to construct models of degree $\leq d$. We make this precise by introducing the notion of *fundamental models*, from which all other models can be constructed. Since there are finitely many fundamental models of degree $\leq d$, we are satisfied with our classification if we can find an upper bound for $\deg(\mathcal{M})$, where \mathcal{M} ranges over all models in Δ_n . This would imply that there are finitely many fundamental models in Δ_n .

Our main theorem gives such an upper bound for models contained in Δ_2 , Δ_3 , and Δ_4 .

Theorem 1.1. *Let $n \leq 4$ and let $\mathcal{M} \subseteq \Delta_n$ be a one-dim. discrete model with ML degree one. Then*

$$\deg(\mathcal{M}) \leq 2n - 1.$$

To prove Theorem 1.1 we use a strategy inspired by the literature on chip-firing [5], which motivates the formulation of an equivalent combinatorial problem. We observe that our one-dimensional models admit a parametrization

$$p: [0, 1] \rightarrow \Delta_n, \quad t \mapsto (w_{\nu} t^{i_{\nu}} (1-t)^{j_{\nu}})_{\nu=0}^n$$

which enables us to represent models as sets of integers on a grid. For instance, the model in Δ_2 parametrized by $p(t) = (t^2, 2t(1-t), (1-t)^2)$ can be represented by the following picture.

$$\begin{array}{c} 1 \\ \cdot 2 \\ -1 \cdot 1 \end{array}$$

In such a picture, the grid point with coordinates (i, j) represents the monomial $t^i(1-t)^j$. The integer entry at that point represents the coefficient of that monomial in the parametrization, where a dot represents the entry 0. The entry -1 at the point $(0, 0)$ indicates that the coordinates of the parametrization add up to 1. We think of these entries as ‘chips’ on the grid, allowing for negative chips. Thus we call such a representation a *chip configuration*.

Any chip on the grid can be split into two further chips, which are then placed directly to the north and to the east of the original chip. We can ‘split a chip’ where there are none by adding a negative chip. Finally, we can unsplit a chip by performing a splitting move in reverse. Starting from the zero configuration, these chipsplitting moves can be used to produce models. For instance, we get the model above by performing chipsplitting moves at $(0, 0)$, $(1, 0)$, and $(0, 1)$, as visualized below.

$$\begin{array}{cccc} \cdot & \cdot & \cdot & 1 \\ \cdot \cdot & 1 \cdot & 1 1 & \cdot 2 \\ 0 \cdot \cdot & -1 1 \cdot & -1 \cdot 1 & -1 \cdot 1 \end{array}$$

In this view, Theorem 1.1 becomes a combinatorial statement about the possible outcomes of these sequences of chipsplitting moves, which we call *chipsplitting games*.

Outline. In Section 2 we explain how to use Theorem 1.1 to classify all models in Δ_n for $n \leq 4$. In Section 3 we introduce chipsplitting games, their basic properties, and formulate the combinatorial equivalent of Theorem 1.1. In Section 4 we explain the connection between models and chipsplitting games. In Sections 5–7 we prove Theorem 1.1 in the language of chipsplitting games for $n \leq 2$, $n = 3$, and $n = 4$, respectively. In Section 8 we discuss examples, computations, and future directions.

Code. We use the computer algebra system Sage [7] to assist us in our proofs, especially in Section 7, and to implement our algorithm for finding fundamental models in Section 8. The code is available on MathRepo at <https://mathrepo.mis.mpg.de/ChipsplittingModels/>.

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2. FUNDAMENTAL MODELS

A *one-dimensional* (parametric, discrete) *algebraic statistical model* is a subset of Δ_n which is the image of a rational map $p: I \rightarrow \Delta_n$ whose components $p_0(t), \dots, p_n(t)$ are rational functions in t , where $I \subseteq \mathbb{R}$ is a union of closed intervals such that $p(\partial I) \subseteq \partial \Delta_n$. Alternatively, such a model can be described as the intersection of Δ_n with a parametrized curve $\{\gamma(t) \mid t \in \mathbb{R}\} \subseteq \mathbb{R}^{n+1}$ with rational entries in the t .

Let $\mathcal{M} \subseteq \Delta_n$ be a one-dimensional algebraic model which is parametrized by the rational functions $p_0(t), \dots, p_n(t)$. The equation $\sum_{\nu} p_{\nu}(t) = 1$ holds for infinitely many and thus for all t . We multiply it by the least common denominator of the $p_{\nu}(t)$ to obtain an equation of the form $\sum_{\nu} a_{\nu}(t) = b(t)$, where $a_0(t), \dots, a_n(t), b(t)$ are polynomials in t . Thus, \mathcal{M} is determined by a collection (a_0, \dots, a_n, b) of polynomials in t satisfying $\sum_{\nu} a_{\nu} = b$. The parametrization of \mathcal{M} is recovered by setting $p_{\nu} = a_{\nu}/b$, where we may assume that the polynomials a_0, \dots, a_n, b share no factor common to all of them.

In maximum likelihood estimation, one seeks to maximize the *log-likelihood* $\ell_u(p) = \sum_{\nu} u_{\nu} \log(p_{\nu})$ given an empirical distribution $u \in \Delta_n$, over all $p \in \mathcal{M}$. This can be accomplished by first finding all

the critical points of ℓ_u . When \mathcal{M} is one-dimensional, finding these critical points amounts to finding the zeros of the derivative $\ell_u(p(t))'$ with respect to t . In our notation, we have

$$\ell_u(p(t))' = \sum_{\nu} u_{\nu} \frac{a'_{\nu}}{a_{\nu}} - \sum_{\nu} u_{\nu} \frac{b'}{b},$$

a rational expression in t which we abbreviate as ℓ'_u . In algebraic statistics, the *maximum likelihood degree* $\text{mld}(\mathcal{M})$ of \mathcal{M} is the number of solutions over \mathbb{C} to this equation for general $u \in \mathbb{C}^n$. In our case, this number can be determined in terms of the roots of the a_{ν} and b , as the next lemma shows.

Lemma 2.1. *Let f be the product of all the distinct complex linear factors occurring among the polynomials a_0, \dots, a_n, b . Then $\text{mld}(\mathcal{M}) = \deg(f) - 1$.*

Proof. Every factor of a polynomial g with multiplicity k occurs in g' with multiplicity $k - 1$. So the expression

$$f\ell'_u = \sum_{\nu} u_{\nu} \frac{fa'_{\nu}}{a_{\nu}} - \sum_{\nu} u_{\nu} \frac{fb'}{b}$$

is a polynomial in t of degree $\deg(f) - 1$. All roots of the rational function ℓ'_u are roots of $f\ell'_u$. It remains to show that no new roots were introduced. That is, that no root of f is also a root of $f\ell'_u$. Thus, let ζ be a complex linear factor of f and $\zeta_0 \in \mathbb{C}$ its derivative. Rewrite $f\ell'_u$ as

$$\sum_{\nu=0}^{n+1} u_{\nu} \frac{fa'_{\nu}}{a_{\nu}}$$

with $a_{\nu+1} := b$ and $u_{n+1} := -\sum_{\nu=0}^n u_{\nu}$. For $\nu = 0, \dots, n+1$, write $a_{\nu} = \zeta^{k_{\nu}} r_{\nu}$ and $f = \zeta r$ such that $\zeta \nmid r_{\nu}, r$. Then for all ν we have $fa'_{\nu}/a_{\nu} = \zeta r k_{\nu} \zeta_0 / \zeta + \zeta r r'_{\nu} / r_{\nu} \equiv \zeta_0 k_{\nu} r \pmod{\zeta}$. Consequently,

$$f\ell'_u \equiv \zeta_0 r \sum_{\nu=0}^{n+1} u_{\nu} k_{\nu} \equiv \zeta_0 r \sum_{\nu=0}^n u_{\nu} (k_{\nu} - k_{n+1}) \pmod{\zeta}.$$

Not all the $(k_{\nu} - k_{n+1})$ for $\nu = 0, \dots, n$ can be zero since ζ is a factor of some a_{ν} for $\nu = 0, \dots, n+1$, but not all of them. Hence, because the u_{ν} are generic we may assume that $\sum_{\nu} u_{\nu} (k_{\nu} - k_{n+1}) \neq 0$. Since additionally $\zeta_0 r \not\equiv 0 \pmod{\zeta}$, we have $f\ell'_u \not\equiv 0 \pmod{\zeta}$, so $\zeta \nmid f\ell'_u$. \square

In this paper we are interested in classifying one-dimensional models of ML degree *one*. The next proposition is the first step in our classification.

Proposition 2.2. *Every one-dimensional discrete model \mathcal{M} of ML degree one has a parametrization of the form*

$$p: [0, 1] \rightarrow \Delta_n, \quad t \mapsto (w_{\nu} t^{i_{\nu}} (1-t)^{j_{\nu}})_{\nu=0}^n$$

for some nonnegative exponents i_{ν}, j_{ν} and positive real coefficients w_{ν} for $\nu = 0, \dots, n$.

Proof. Let \mathcal{M} be defined by the polynomials a_0, \dots, a_n, b with $\sum_{\nu} a_{\nu} = b$. By Lemma 2.1, these polynomials split as products of the same two complex factors. The $n+1$ faces of Δ_n lie on the $n+1$ coordinate hyperplanes of \mathbb{R}^n . Thus, the set I in the parametrization $p: I \rightarrow \mathcal{M}$ is a single closed interval because $p(\partial I) \subseteq \partial \Delta_n$ and the a_{ν}, b have exactly two zeros among them. In particular, these zeros are real and coincide with the endpoints of I . Without changing \mathcal{M} , we may reparametrize and assume that $I = [0, 1]$. We may write

$$\begin{aligned} a_{\nu}(t) &= w_{\nu} t^{i_{\nu}} (1-t)^{j_{\nu}} \\ b(t) &= w t^i (1-t)^j, \end{aligned}$$

for $w_{\nu}, w \in \mathbb{R}_{>0}$ and $i_{\nu}, j_{\nu}, i, j \in \mathbb{Z}_{\geq 0}$ for all ν . If $i > 0$, then $i_{\nu} = 0$ for some ν and we arrive at a contradiction by evaluating the equation $\sum_{\nu} a_{\nu} = b$ at $t = 0$. So $i = 0$. Similarly, we must have $j = 0$. By dividing by w we now arrive at the required form for p . \square

Thus, our goal is to provide a classification of the parametrizations of models specified by Proposition 2.2. For brevity, we may refer to these simply as ‘models’ from now on. We will show how these models can be built up from progressively simpler models, the simplest of which we will call ‘fundamental models’.

Proposition 2.2 shows that every model $\mathcal{M} \subseteq \Delta_n$ can be represented by a finite sequence $(w_\nu, i_\nu, j_\nu)_{\nu=0}^n$ for some nonnegative exponents i_ν, j_ν and positive real coefficients w_ν . The degree of \mathcal{M} as an algebraic variety, denoted by $\deg(\mathcal{M})$, is $\max\{\deg(i_\nu, j_\nu) \mid \nu \in \{0, \dots, n\}\}$ where $\deg(i, j) := i + j$.

We consider two models in Δ_n equivalent if they differ only by a relabeling of the coordinates on Δ_n . Although $(w_\nu, i_\nu, j_\nu)_{\nu=0}^n$ and $(w_\nu, j_\nu, i_\nu)_{\nu=0}^n$ represent the same subset of Δ_n , we shall count these two representations as distinct ‘models’ unless they are equal up to reordering. These two models differ by the reparametrization $t \mapsto p(t-1)$.

We now define our first simpler subclass of the class of models.

Definition 2.3. A model represented by $(w_\nu, i_\nu, j_\nu)_{\nu=0}^n$ is *reduced* if the exponent pairs (i_ν, j_ν) are not equal to $(0, 0)$ and pairwise distinct.

Proposition 2.4. *Every one-dimensional discrete model of ML degree one is the image of a reduced model under a chain of linear embeddings of the form*

$$(1) \quad \Delta_{n-1} \rightarrow \Delta_n, \quad (p_0, \dots, \hat{p}_\nu, \dots, p_n) \mapsto (\lambda p_0, \dots, 1 - \lambda, \dots, \lambda p_n), \quad \lambda \in [0, 1]$$

or

$$(2) \quad \Delta_{n-1} \rightarrow \Delta_n, \quad (p_0, \dots, p_\nu, \dots, \hat{p}_\mu, \dots, p_n) \mapsto (p_0, \dots, \lambda p_\nu, \dots, (1 - \lambda)p_\nu, \dots, p_n), \quad \lambda \in [0, 1].$$

Proof. Let $(w_\nu, i_\nu, j_\nu)_{\nu=0}^n$ represent a model \mathcal{M} . If $(i_\nu, j_\nu) = (0, 0)$ for some ν then $w_\nu < 1$. Let $\lambda := 1 - w_\nu$. Then \mathcal{M} is the image under the linear embedding (1) of the model represented by

$$(w_\nu/(1 - w_\nu), i_\nu, j_\nu)_{i=0, i \neq \nu}^n.$$

Similarly, suppose that $(i_\nu, j_\nu) = (i_\mu, j_\mu)$ for some $\nu \neq \mu$ and let $\lambda := w_\nu/(w_\nu + w_\mu)$. Then \mathcal{M} is the image under the linear embedding (2) of the model represented by

$$(w_\nu + \delta_{i\nu} w_\mu, i_\nu, j_\nu)_{i=0, i \neq \mu}^n. \quad \square$$

Remark 2.5. If Δ_n contains a model of degree d , then $\Delta_{n'}$ must contain a reduced model of degree d for some $n' \leq n$. Therefore, to prove Theorem 1.1 it is enough to consider reduced models only.

Definition 2.6. A reduced model represented by $(w_\nu, i_\nu, j_\nu)_{\nu=0}^n$ is a *fundamental model* if, given the exponents (i_ν, j_ν) , the weights (w_ν) are uniquely determined by the constraint $\sum_\nu p_\nu = 1$.

Thus, for any given set of exponents (i_ν, j_ν) , we can check whether there is a fundamental model with these exponents by solving a system of affine-linear equations in the weights w_ν . Similarly, the set of models with these fixed exponents is always an affine-linear half space of dimension at most $n + 1$.

Example 2.7. Consider the sequence of exponents $((2, 0), (1, 1), (0, 2))$. The polynomial constraint $w_0 t^2 + w_1 t(1 - t) + w_2(1 - t)^2 = 1$ leads to the affine-linear system $w_0 - w_1 + w_2 = 0$, $w_1 - 2w_2 = 0$, $w_2 - 1 = 0$. The unique solution $(1, 2, 1)$ defines the fundamental model $t \mapsto (t^2, 2t(1 - t), (1 - t)^2)$.

We shall now see that every reduced model can be constructed from finitely many fundamental models in a finite number of steps. For this, we represent a reduced model by the function $f: \mathbb{Z}^2 \rightarrow \mathbb{R}_{\geq 0}$ that sends an exponent pair (i_ν, j_ν) to its associated coefficient w_ν . We call the set of exponent pairs (i_ν, j_ν) the *support* of \mathcal{M} . It equals $\text{supp}(f)$.

Definition 2.8. Let \mathcal{M}_1 and \mathcal{M}_2 be reduced models represented by the functions $f_1, f_2: \mathbb{Z}^2 \rightarrow \mathbb{R}_{\geq 0}$. Let $0 < \mu < 1$. The *composite* $\mathcal{M}_1 *_\mu \mathcal{M}_2$ of \mathcal{M}_1 and \mathcal{M}_2 is the reduced model represented by

$$g: \mathbb{Z}^2 \rightarrow \mathbb{R}_{\geq 0}, \quad g(i, j) := \mu f_1(i, j) + (1 - \mu) f_2(i, j).$$

Proposition 2.9. *Every reduced model is the composite of finitely many fundamental models.*

Proof. Let \mathcal{M} be a reduced model represented by $(w_\nu, i_\nu, j_\nu)_{\nu=0}^n$. If $n \leq 1$ then \mathcal{M} is fundamental. So, let $n \geq 2$ and \mathcal{M} not fundamental. It suffices to show that \mathcal{M} is the composite of two models whose supports are proper subsets of S . Since \mathcal{M} is not fundamental, there exist $x_0, \dots, x_n \in \mathbb{R}$, not all zero, such that $\sum_{\nu=0}^n x_\nu t^{i_\nu} (1-t)^{j_\nu} = 0$. Since this equality holds for all $t \in (0, 1)$, we have at least one positive and one negative x_ν . Let

$$\lambda := \min\{w_\nu/|x_\nu| \mid \nu \in \{0, \dots, n\}, x_\nu < 0\}, \quad u_\nu := w_\nu + \lambda x_\nu \text{ for } \nu \in \{0, \dots, n\},$$

and $S_1 := \{(i_\nu, j_\nu) \mid \nu \in \{0, \dots, n\}, u_\nu \neq 0\}$. Then we have $\lambda > 0$ and $u_\nu \geq 0$ for all $\nu \in \{0, \dots, n\}$, the latter of which we verify by distinguishing between the cases $x_\nu \geq 0$ and $x_\nu < 0$. For all ν we have $u_\nu = 0$ if and only if $x_\nu < 0$ and $\lambda = w_\nu/|x_\nu|$. Thus S_1 is a nonempty proper subset of S . Since $\sum_{\nu=0}^n u_\nu s_\nu = 1$, the coefficients u_ν for $(i_\nu, j_\nu) \in S_1$ define a reduced model \mathcal{M}_1 with support S_1 . Let

$$\mu := \min\{w_\nu/u_\nu \mid \nu \in \{0, \dots, n\}, u_\nu \neq 0\}, \quad v_\nu := (w_\nu - \mu u_\nu)/(1 - \mu) \text{ for } \nu \in \{0, \dots, n\},$$

and $S_2 := \{(i_\nu, j_\nu) \mid \nu \in \{0, \dots, n\}, v_\nu \neq 0\}$. Then $\mu > 0$. Since at least one of the x_ν is positive, we have $u_\nu > w_\nu$ for some ν , and thus $\mu < 1$. We have $v_\nu \geq 0$ by the definition of μ and $v_\nu = 0$ if and only if $u_\nu \neq 0$ and $\mu = w_\nu/u_\nu$. Thus S_2 is a nonempty proper subset of S and we have $S_1 \cup S_2 = S$. Since $\sum_{\nu=0}^n v_\nu x_\nu = 1$, the coefficients v_ν for $(i_\nu, j_\nu) \in S_2$ define a reduced model \mathcal{M}_2 with support S_2 . We conclude by noting that $w_\nu = \mu u_\nu + (1 - \mu)v_\nu$ for all $\nu \in \{0, \dots, n\}$. Thus, $\mathcal{M} = \mathcal{M}_1 *_\mu \mathcal{M}_2$. \square

Remark 2.10. If a reduced model $\mathcal{M} \subseteq \Delta_n$ is not fundamental, then by Proposition 2.9 there exists $n' < n$ and a fundamental model in $\Delta_{n'}$ of the same degree as \mathcal{M} . Thus, it suffices to prove Theorem 1.1 for fundamental \mathcal{M} . In turn, Theorem 1.1 implies that there are only finitely many fundamental models in Δ_n for $n \leq 4$. This is because for all d there are only finitely many subsets $S \subseteq \mathbb{Z}^2$ that can be the support of a fundamental model \mathcal{M} of degree d , and S determines \mathcal{M} uniquely.

Our classification of one-dimensional discrete models of ML degree one is now complete. We summarize it in Theorem 2.11, all elements of which we already established in this section. Part (c) uses Theorem 1.1, which we will prove in Sections 5–7. We visualize our classification in Figure 1.

Theorem 2.11.

- (a) Every one-dimensional discrete model of ML degree one $\mathcal{M} \subseteq \Delta_n$ is the image of a reduced model $\mathcal{M}' \subseteq \Delta_{n'}$ under a linear embedding $\Delta_{n'} \rightarrow \Delta_n$ for some $n' \leq n$.
- (b) Every reduced model $\mathcal{M}' \subseteq \Delta_{n'}$ can be written as the composite

$$\mathcal{M}' = \mathcal{M}_1 *_{\mu_1} (\dots *_{\mu_{m-1}} (\mathcal{M}_{m-1} *_{\mu_m} \mathcal{M}_m) \dots)$$

of finitely many fundamental models $\mathcal{M}_1, \dots, \mathcal{M}_m$.

- (c) For $n \leq 4$, there are only finitely many fundamental models in Δ_n . \square

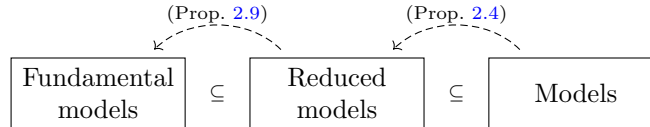


FIGURE 1. A classification of one-dimensional discrete models of ML degree one.

Example 2.12. Let us classify all one-dimensional models \mathcal{M} of ML degree one in the triangle Δ_2 , up to coordinate permutations. The unique model \mathcal{M}_0 in Δ_1 is parametrized by $t \mapsto (t, (1-t))$. Since $\mathcal{M}_0 *_\mu \mathcal{M}_0 = \mathcal{M}_0$, all models in Δ_2 are either fundamental or non-reduced. Theorem 1.1 gives a bound for the algebraic degree of \mathcal{M} : we have $\deg(\mathcal{M}) \leq 3$. Hence, to find all fundamental models we check all possible sets of exponent pairs (or *supports*) $S \subseteq \{(i, j) \mid 0 < i + j \leq 3\}$ of size $n + 1 = 3$. We report the results in Figure 2.

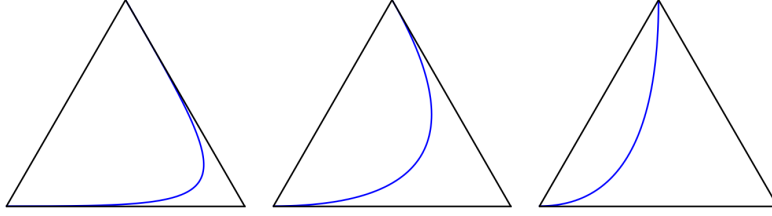


FIGURE 2. Fundamental models in Δ_2 . These correspond to the parametrizations $t \mapsto ((1-t)^3, 3t(1-t), t^3)$, $t \mapsto ((1-t)^2, 2(1-t)t, t^2)$, and $t \mapsto ((1-t), t(1-t), t^2)$, from left to right. Their supports are $\{(0,3), (1,1), (3,0)\}$, $\{(0,2), (1,1), (2,0)\}$, and $\{(0,1), (1,1), (2,0)\}$, respectively. In Δ_2 there is a further fundamental model with support $\{(0,2), (1,0), (1,1)\}$, but it is identical to the third model in this picture after a permutation of the coordinates of Δ_n and the reparametrization $t \mapsto 1-t$.

As for non-reduced models, there are up to coordinate permutations only two linear embeddings $\Delta_1 \rightarrow \Delta_2$ of the form (1) or (2) that can be used to construct \mathcal{M} from \mathcal{M}_0 . These can vary with the parameter λ and are reported in Figure 3 for $\lambda = 1/3$.

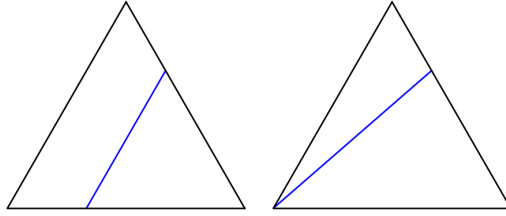


FIGURE 3. Non-reduced models in Δ_2 . These arise from linear embeddings $\Delta_1 \rightarrow \Delta_2$ of type (1) and (2), respectively. They are given by $t \mapsto ((1-\lambda)t, \lambda, (1-\lambda)(1-t))$ and $t \mapsto ((1-t), \lambda t, (1-\lambda)t)$, where $\lambda := 1/3$. All other non-reduced one-dimensional models of ML degree one in Δ_2 arise from these two by varying λ and permuting the coordinates of Δ_2 .

3. CHIPSPLITTING GAMES

In this section we lay some groundwork for proving Theorem 1.1. We begin with the general definition of a (directed) chipsplitting game. Let (V, E) be a directed graph without loops.

Definition 3.1. Let $V' \subseteq V$ be the subset of vertices with ≥ 1 outgoing edge.

- (a) A *chip configuration* is a vector $w = (w_v)_{v \in V} \in \mathbb{Z}^V$ such that $\#\{v \in V \mid w_v \neq 0\} < \infty$.
- (b) The *initial configuration* is the zero vector $0 \in \mathbb{Z}^V$.
- (c) A *splitting move* at $p \in V$ maps a chip configuration $w = (w_v)_{v \in V}$ to the chip configuration $\tilde{w} = (\tilde{w}_v)_{v \in V}$ defined by

$$\tilde{w}_v := \begin{cases} w_v - 1 & \text{if } v = p, \\ w_v + 1 & \text{if } (p, v) \in E, \text{ i.e., } E \text{ contains an edge from } p \text{ to } v, \\ w_v & \text{otherwise.} \end{cases}$$

An *unsplitting move* at p maps \tilde{w} back to w .

- (d) A *chipsplitting game* f is a finite sequence of splitting and unsplitting moves. The *outcome* of f is the chip configuration obtained from the initial configuration after executing all the moves in f .
- (e) A (*chipsplitting*) *outcome* is the outcome of any chipsplitting game.

Note that the moves in our game are all reversible and commute with each other. In particular, the order of the moves in a game does not matter and every outcome is the outcome of a game such that at no point in V both a splitting and an unsplitting move occurs. We call games that have this property *reduced*. We usually assume chipsplitting games are reduced. The map

$$\begin{aligned} \{\text{reduced chipsplitting games on } (V, E)\} / \sim &\rightarrow \{g: V' \rightarrow \mathbb{Z} \mid \#\{p \in V' \mid g(p) \neq 0\} < \infty\} \\ f &\mapsto (p \mapsto \text{number of moves at } p \text{ in } f) \end{aligned}$$

is a bijection, where we count unsplitting moves negatively and consider two games f, g equivalent if they are the same up to reordering. We identify a reduced chipsplitting game f with its corresponding function $V' \rightarrow \mathbb{Z}$. The outcome $w = (w_v)_{v \in V}$ of f now satisfies

$$w_v = -f(v) + \sum_{\substack{p \in V' \\ (p, v) \in E}} f(v),$$

where we write $f(v) = 0$ when $v \notin V'$.

Remark 3.2. Let A be an abelian group. The definitions above naturally extend from \mathbb{Z} to A , i.e., to the setting where the number of chips at a point and number of times a move is repeated are both allowed to be any element of A . Here (resp. when $A = \mathbb{Q}, \mathbb{R}$), we say that the chip configurations, chipsplitting games and outcomes are A -valued (resp. *rational, real*).

We now define the directed graphs (V_d, E_d) we consider in this paper. For $d \in \mathbb{N} \cup \{\infty\}$, write

$$\begin{aligned} V_d &:= \{(i, j) \in \mathbb{Z}_{\geq 0}^2 \mid \deg(i, j) \leq d\}, \\ E_d &:= \{(v, v + e) \mid v \in V_{d-1}, e \in \{(1, 0), (0, 1)\}\}, \end{aligned}$$

where $\deg(i, j) := i + j$ is the *degree* of $(i, j) \in \mathbb{Z}_{\geq 0}^2$.

Example 3.3. We depict a chip configuration $w = (w_{i,j})_{(i,j) \in V_d} \in \mathbb{Z}^{V_d}$ as a triangle of numbers with $w_{i,j}$ being the number in the i th column from the left and j th row from the bottom.

$$\begin{array}{cccccccc} \cdot & & & & & 1 & & 1 & & 1 \\ \cdot & & & & & \cdot & 1 & & \cdot & 1 & & \cdot \\ \cdot & & & & & 1 & \cdot & & & \cdot & 2 & & \cdot \\ \cdot & & & & & 1 & 1 & & & \cdot & 2 & & \cdot \\ 0 & & & & & -1 & \cdot & 1 & & & -1 & \cdot & 1 & & & -1 & \cdot & 1 & & & -1 & \cdot & 1 & & & -1 & \cdot & 1 \end{array}$$

When $w_{i,j} = 0$, we usually write \cdot at position (i, j) instead of 0. In the examples above, we have $d = 3$. The leftmost configuration is the initial configuration. From left to right, we obtain the next five configurations by successively executing splitting moves at $(0, 0)$, $(1, 0)$, $(0, 1)$, $(0, 2)$, and $(2, 0)$, respectively. Finally, we obtain the rightmost configuration by applying an unsplitting move at $(1, 1)$.

Remark 3.4. The notion of chipsplitting games is inspired by that of chipfiring games. For a thorough treatment of the latter, see [5]. In fact, by using powers of two one can prove that our chipsplitting games are equivalent to certain chipfiring games, provided the latter allow ‘unfiring’, or reversing a firing move. All notions of Definition 3.1 have chipfiring equivalents. In this paper, we use chipsplitting games as they relate more directly to the statistical models of Section 2.

Definition 3.5. Let $w = (w_{i,j})_{(i,j) \in V_d}$ be a chip configuration.

- (a) The *positive support* of w is $\text{supp}^+(w) := \{(i, j) \in V_d \mid w_{i,j} > 0\}$.
- (b) The *negative support* of w is $\text{supp}^-(w) := \{(i, j) \in V_d \mid w_{i,j} < 0\}$.
- (c) The *support* of w is $\text{supp}(w) := \{(i, j) \in V_d \mid w_{i,j} \neq 0\} = \text{supp}^+(w) \cup \text{supp}^-(w)$.
- (d) The *degree* of w is $\deg(w) := \max\{\deg(i, j) \mid (i, j) \in \text{supp}(w)\}$.
- (e) We say that w is *valid* when $\text{supp}^-(w) \subseteq \{(0, 0)\}$.
- (f) We say that w is *weakly valid* when for all $(i, j) \in \text{supp}^-(w)$ one of the following holds:
 - (i) $0 \leq i, j \leq 3$,
 - (ii) $0 \leq i \leq 3$ and $\deg(i, j) \geq d - 3$, or
 - (iii) $0 \leq j \leq 3$ and $\deg(i, j) \geq d - 3$.

Figure 4 illustrates the notion of a weakly valid outcome, which will first be used in Section 6.5.

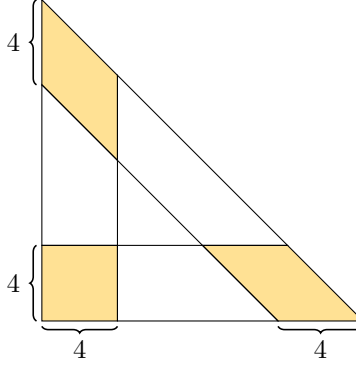


FIGURE 4. The corners of the four-entries wide outer ring of the triangle V_d . A chip configuration is weakly valid if its negative support is contained in the orange area.

We can now state our main result in the language of valid outcomes.

Theorem 3.6. *Let $n \leq 4$ and let w be a valid outcome with a positive support of size $n + 1$. Then*

$$\deg(w) \leq 2n - 1.$$

We will prove Theorem 3.6 in Sections 5–7 (See Theorems 5.14, 6.20 and 7.2). In the rest of this section, we collect results about chipsplitting outcomes that will help us later.

3.1. Symmetry. For every $d \in \mathbb{N} \cup \{\infty\}$, define an action of the group $S_2 = \langle (12) \rangle$ on \mathbb{Z}^{V_d} by setting

$$(12) \cdot (w_{i,j})_{(i,j) \in V_d} := (w_{j,i})_{(i,j) \in V_d},$$

where clearly $(12) \cdot ((12) \cdot w) = w$ for all $w \in \mathbb{Z}^{V_d}$. We also let S_2 act on V_d by $(12) \cdot (i, j) := (j, i)$.

The initial configuration is fixed by S_2 . Let $w \in \mathbb{Z}^{V_d}$, $p \in V_{d-1}$, and let \tilde{w} be the result of applying an (un)splitting move at p to w . Then $(12) \cdot \tilde{w}$ is the result of applying an (un)splitting move at $(12) \cdot p$ to $(12) \cdot w$. So we see that if w is the outcome of a reduced chipsplitting game f , then $(12) \cdot w$ is the outcome of the chipsplitting game $(i, j) \mapsto f(j, i)$. Hence the space of outcomes is closed under the action of S_2 . Let $w \in \mathbb{Z}^{V_d}$ be a chip configuration. Then

$$\begin{aligned} \text{supp}^+((12) \cdot w) &= (12) \cdot \text{supp}^+(w), & \text{supp}^-((12) \cdot w) &= (12) \cdot \text{supp}^-(w), \\ \text{supp}((12) \cdot w) &= (12) \cdot \text{supp}(w), & \deg((12) \cdot w) &= \deg(w). \end{aligned}$$

Furthermore, w is (weakly) valid if and only if $(12) \cdot w$ is (weakly) valid.

3.2. Pascal equations. Another way to study the space of outcomes is via the set of linear forms that vanish on it. A *linear form* on \mathbb{Z}^{V_d} is a function $\mathbb{Z}^{V_d} \rightarrow \mathbb{Z}$ of the form

$$(w_{i,j})_{(i,j) \in V_d} \mapsto \sum_{(i,j) \in V_d} c_{i,j} w_{i,j},$$

which we will denote by $\sum_{(i,j) \in V_d} c_{i,j} x_{i,j}$. The group S_2 acts on the space of linear forms on \mathbb{Z}^{V_d} via

$$(12) \cdot \sum_{(i,j) \in V_d} c_{i,j} x_{i,j} := \sum_{(i,j) \in V_d} c_{j,i} x_{i,j}.$$

Definition 3.7. We say that a linear form $\sum_{(i,j) \in V_d} c_{i,j} x_{i,j}$ is a *Pascal equation* when

$$c_{i,j} = c_{i+1,j} + c_{i,j+1}$$

for all $(i, j) \in V_{d-1}$.

This terminology is inspired by the Pascal triangle, whose entries satisfy the same condition. The space of Pascal equations is closed under the action of S_2 .

Proposition 3.8. *Let $(a_k)_{k=0}^d \in \mathbb{Z}^{\mathbb{N}^{\leq d}}$ be any vector.*

- (a) *There exists a unique Pascal equation $\sum_{(i,j) \in V_d} c_{i,j} x_{i,j}$ such that $c_{0,j} = a_j$ for all $0 \leq j \leq d$.*
- (b) *There exists a unique Pascal equation $\sum_{(i,j) \in V_d} c_{i,j} x_{i,j}$ such that $c_{i,0} = a_i$ for all $0 \leq i \leq d$.*

Proof. (a) Set $c_{0,j} := a_j$ for all integers $0 \leq j \leq d$ and define

$$c_{i+1,j} := c_{i,j} - c_{i,j+1}$$

for all $(i,j) \in V_d$ via recursion on $i > 0$. Then $\sum_{(i,j) \in V_d} c_{i,j} x_{i,j}$ is a Pascal equation such that $c_{0,j} = a_j$ for all integers $0 \leq j \leq d$. Clearly, it is the only Pascal equation with this property.

(b) Write

$$(12) \cdot \sum_{(i,j) \in V_d} c_{i,j} x_{i,j} = \sum_{(i,j) \in V_d} d_{i,j} x_{i,j}.$$

Then $c_{k,0} = a_k$ if and only if $d_{0,k} = a_k$ and hence the statement follows from (a). \square

Our next goal is to prove that a chip configuration is an outcome if and only if all Pascal equations vanish at it.

Proposition 3.9. *Let $w \in \mathbb{Z}^{V_d}$ be a chip configuration. Then the value at w of any given Pascal equation on \mathbb{Z}^{V_d} is invariant under (un)splitting moves. In particular, all Pascal equations on \mathbb{Z}^{V_d} vanish at all outcomes.*

Proof. Let $w = (w_{i,j})_{(i,j) \in V_d}$ be a chip configuration and suppose we obtain $\tilde{w} = (\tilde{w}_{i,j})_{(i,j) \in V_d}$ from w by applying a chipsplitting move at $(i', j') \in V_{d-1}$. Let $\sum_{(i,j) \in V_d} c_{i,j} x_{i,j}$ be a Pascal equation. Then we see that

$$\sum_{(i,j) \in V_d} c_{i,j} \tilde{w}_{i,j} = \sum_{(i,j) \in V_d} c_{i,j} \begin{cases} w_{i,j} - 1 & \text{if } (i,j) = (i', j'), \\ w_{i,j} + 1 & \text{if } (i,j) = (i' + 1, j), \\ w_{i,j} + 1 & \text{if } (i,j) = (i', j' + 1), \\ w_{i,j} & \text{otherwise} \end{cases} = \sum_{(i,j) \in V_d} c_{i,j} w_{i,j}$$

since $c_{i'+1, j'} + c_{i', j'+1} - c_{i', j'} = 0$, which proves the first claim. For the second claim it suffices to note that all Pascal equations vanish at the initial configuration. \square

Let $w = (w_{i,j})_{(i,j) \in V_d}$ be a degree- e chip configuration. Then there exists a unique reduced chipsplitting game that uses only moves at $(i,j) \in V_{d-1}$ with $\deg(i,j) = e - 1$ and that sets the values $w_{0,e}, w_{1,e-1}, \dots, w_{e-1,1}$ to 0. Note that these moves do not alter the alternating sum $\sum_{k=0}^e (-1)^k w_{k,e-k}$. So, if $\sum_{k=0}^e (-1)^k w_{k,e-k} = 0$, this chipsplitting game also sets $w_{e,0}$ to 0. This motivates the following definition.

Definition 3.10. Let $w = (w_{i,j})_{(i,j) \in V_d}$ be a degree- e chip configuration such that

$$\sum_{k=0}^e (-1)^k w_{k,e-k} = 0.$$

The *retraction* of w is the unique chip configuration obtained from w using moves at points $(i,j) \in V_{d-1}$ with $\deg(i,j) = e - 1$ such that $\deg(w) < e$.

Proposition 3.11. *Let $w = (w_{i,j})_{(i,j) \in V_d}$ be a degree- e chip configuration. Then w is an outcome if and only if $\sum_{k=0}^e (-1)^k w_{k,e-k} = 0$ and the retraction of w is an outcome.*

Proof. If $\sum_{k=0}^e (-1)^k w_{k,e-k} = 0$, then w and its retraction are obtained from each other using finite sequences of moves. So it suffices to prove that $\sum_{k=0}^e (-1)^k w_{k,e-k} = 0$ holds when w is an outcome. Assume that w is the outcome of a reduced chipsplitting game f . Then $e - 1$ is the maximal degree of a point in V_{d-1} at which a move in f occurred. As moves at (i,j) preserve the value of $\sum_{k=0}^e (-1)^k w_{k,e-k}$ for all $(i,j) \in V_{d-1}$ with $\deg(i,j) \leq e - 1$, we see that $\sum_{k=0}^e (-1)^k w_{k,e-k} = 0$. \square

Proposition 3.12. *Let $w \in \mathbb{Z}^{V_d}$ be a chip configuration and suppose that all Pascal equations on \mathbb{Z}^{V_d} vanish at w . Then w is an outcome.*

Proof. By Proposition 3.8, for every integer $0 \leq e \leq d$ there exists a Pascal equation

$$\phi^{(e)} := \sum_{(i,j) \in V_d} c_{i,j}^{(e)} x_{i,j}$$

with $c_{0,j}^{(e)} = 0$ for $j < e$ and $c_{0,e}^{(e)} = 1$. Note that $c_{i,j}^{(e)} = 0$ for all $(i,j) \in V_d$ with $\deg(i,j) < e$ and $c_{k,e-k}^{(e)} = (-1)^k$ for $k \in \{0, \dots, e\}$. Next, note that for $e = \deg(w)$ we have

$$\sum_{k=0}^e (-1)^k w_{i,j} = \phi^{(e)}(w) = 0$$

and hence w has a retraction w' , at which all Pascal equations also vanish. Repeating the same argument, we see that w' also has a retraction w'' , at which all Pascal equations again vanish. After repeating this $e + 1$ times, we arrive at a chip configuration of degree < 0 , which must be the initial configuration. Hence by Proposition 3.11, we see that w is an outcome. \square

A chip configuration $w \in \mathbb{Z}^{V_d}$ is an outcome if and only if all Pascal equations vanish at w . In particular using a larger or smaller V'_d for the same w , provided $d' \geq \deg(w)$, does not change the fact that w is a chipsplitting outcome. Later in this section we see however that fixing a finite d is useful as it provides an additional basis to the space of Pascal equations.

Definition 3.13. Let $0 \leq k \leq d$ be an integer.

- (a) We write ψ_k for the unique Pascal equation $\sum_{(i,j) \in V_d} c_{i,j} x_{i,j}$ such that $c_{0,j} = \delta_{jk}$
- (b) We write $\bar{\psi}_k := (12) \cdot \psi_k$.

Proposition 3.14.

- (a) *We have*

$$\psi_k = (-1)^k \sum_{(i,j) \in V_d} (-1)^j \binom{i}{k-j} x_{i,j} \text{ and } \bar{\psi}_k = (-1)^k \sum_{(i,j) \in V_d} (-1)^i \binom{j}{k-i} x_{i,j}$$

for all integers $0 \leq k \leq d$.

- (b) *Every Pascal equation can be written uniquely as*

$$\sum_{k=0}^d a_k \psi_k = \sum_{k=0}^d b_k \bar{\psi}_k,$$

where $a_k, b_k \in \mathbb{Z}$. When $d < \infty$, the ψ_k and $\bar{\psi}_k$ form two bases of the space of Pascal equations.

Proof. (a) We have $(-1)^{k+j} \binom{0}{k-j} = \delta_{jk}$ and so it suffices to prove that

$$\sum_{(i,j) \in V_d} (-1)^j \binom{i}{k-j} x_{i,j}$$

is in fact a Pascal equation. Indeed, we have

$$(-1)^j \binom{i}{k-j} = (-1)^j \binom{i+1}{k-j} + (-1)^{j+1} \binom{i}{k-(j+1)}$$

for all $(i,j) \in V_d$ as $\binom{a+1}{b+1} = \binom{a}{b+1} + \binom{a}{b}$ for all integers a, b .

- (b) Write

$$\sum_{(i,j) \in V_d} c_{i,j} x_{i,j} = \sum_{k=0}^d a_k \psi_k = \sum_{k=0}^d b_k \bar{\psi}_k.$$

for all $(i, j) \in V_d$ with $\deg(i, j) = d$. So it suffices to show that

$$\sum_{(i,j) \in V_d} \binom{d - (i + j)}{a - i} x_{i,j}$$

is a Pascal equation. Indeed, we have

$$\binom{d - (i + j)}{a - i} = \binom{d - (i + 1 + j)}{a - (i + 1)} + \binom{d - (i + j + 1)}{a - i}$$

for all $(i, j) \in V_{d-1}$ as $\binom{a+1}{b+1} = \binom{a}{b+1} + \binom{a}{b}$ for all integers a, b .

(b) Every Pascal equation can be uniquely written as

$$\sum_{(i,j) \in V_d} c_{i,j} x_{i,j} = \sum_{(a,b) \in V_d \setminus V_{d-1}} c_{a,b} \varphi_{a,b}.$$

So we see that the $\varphi_{a,b}$ form a basis for the space of all Pascal equations. □

Example 3.19. For $d = 7$ and $(a, b) = (3, 4)$, the Pascal equation $\varphi_{a,b}$ can be visualised by writing the coefficients $c_{i,j}$ on the grid V_d as follows:

.						
.	.					
.	.	.				
1	1	1	1			
4	3	2	1	.		
10	6	3	1	.	.	
20	10	4	1	.	.	.
35	15	5	1	.	.	.

We note that the coefficients form a Pascal triangle.

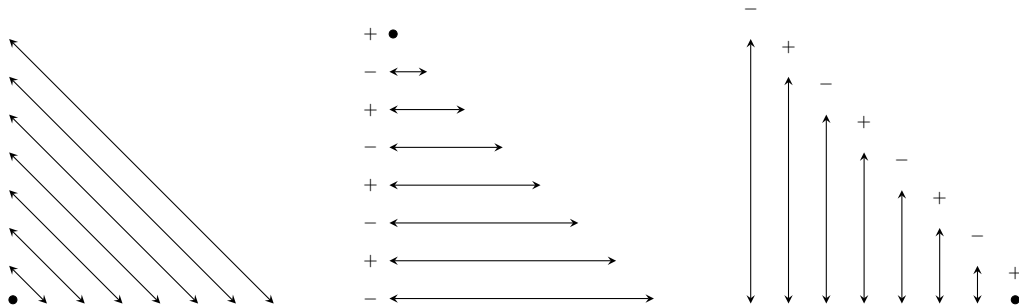
Next we define an action of $S_3 = \langle (12), (123) \rangle$ on \mathbb{Z}^{V_d} . We set

$$\begin{aligned} (12) \cdot (w_{i,j})_{(i,j) \in V_d} &:= (w_{j,i})_{(i,j) \in V_d} =: w^{(a)} \\ (123) \cdot (w_{i,j})_{(i,j) \in V_d} &:= ((-1)^{d-j} w_{j,d-\deg(i,j)})_{(i,j) \in V_d} =: w^{(b)} \end{aligned}$$

for all $w = (w_{i,j})_{(i,j) \in V_d} \in \mathbb{Z}^{V_d}$. It is a routine computation to verify that this determines a well-defined action of S^3 . Under this action, we have

$$\begin{aligned} (13) \cdot (w_{i,j})_{(i,j) \in V_d} &= ((-1)^{d-j} w_{d-\deg(i,j),j})_{(i,j) \in V_d} \\ (23) \cdot (w_{i,j})_{(i,j) \in V_d} &= ((-1)^{d-i} w_{i,d-\deg(i,j)})_{(i,j) \in V_d}, \\ (132) \cdot (w_{i,j})_{(i,j) \in V_d} &= ((-1)^{d-i} w_{d-\deg(i,j),i})_{(i,j) \in V_d}. \end{aligned}$$

The way (12), (13) and (23) act is visualized below. The permutation (12) switches the order of all entries of the same degree. The permutation (13) switches the order of all entries of the same row and changes the signs of alternating rows. The permutation (23) acts similarly on columns.



Note that the set of weakly valid chip configurations is closed under the action of S_3 . The same is true for the space of outcomes.

Proposition 3.20. *The space of outcomes is closed under the action of S_3 .*

Proof. Let $w = (w_{i,j})_{(i,j) \in V_d}$ be an outcome. We already know that $(12) \cdot w$ is again an outcome. So it suffices to prove that $(123) \cdot w$ is an outcome as well. This is indeed the case since

$$\begin{aligned} \psi_k((123) \cdot w) &= (-1)^k \sum_{(i,j) \in V_d} (-1)^j \binom{i}{k-j} (-1)^{d-j} w_{j,d-\deg(i,j)} \\ &= (-1)^{d-k} \sum_{(i',j') \in V_d} \binom{d-(i'+j')}{k-i'} w_{i',j'} \\ &= (-1)^{d-k} \varphi_{k,d-k}(w) = 0 \end{aligned}$$

for all integers $0 \leq k \leq d$. □

We also define an action of S_3 on V_d . We set

$$\begin{aligned} (12) \cdot (i, j) &:= (j, i), & (123) \cdot (i, j) &:= (d - \deg(i, j), i), \\ (13) \cdot (i, j) &:= (d - \deg(i, j), j), & (132) \cdot (i, j) &:= (j, d - \deg(i, j)), \\ (23) \cdot (i, j) &:= (i, d - \deg(i, j)), \end{aligned}$$

for all $(i, j) \in V_d$. We have $\sigma \cdot \text{supp}(w) = \text{supp}(\sigma \cdot w)$ for all $w \in \mathbb{Z}^{V_d}$ and $\sigma \in S_3$.

3.4. Valid outcomes. In this paper, we are mostly interested in valid outcomes, since they correspond to reduced models as explained in Section 4.

Lemma 3.21. *Let $w = (w_{i,j})_{(i,j) \in V_d} \in \mathbb{Z}^{V_d}$ be an outcome and suppose that $\text{supp}^-(w) = \emptyset$. Then w is the initial configuration.*

Proof. We may assume that $d < \infty$. We have $w_{i,j} \geq 0$ for all $(i, j) \in V_d$. For every $(a, b) \in V_d$ of degree d , the equation $\varphi_{a,b}(w) = 0$ shows that $w_{i,j} = 0$ for all $i \in \{0, \dots, a\}$ and $j \in \{0, \dots, b\}$. Combined, this shows that $w_{i,j} = 0$ for all $(i, j) \in V_d$. □

Proposition 3.22. *Let $w = (w_{i,j})_{(i,j) \in V_d} \in \mathbb{Z}^{V_d}$ be an outcome and suppose that $\#\text{supp}^-(w) = 1$. Write $c_0 = \min\{i \mid (i, j) \in V_d \mid w_{i,j} \neq 0\}$, $r_0 = \min\{j \mid (i, j) \in V_d \mid w_{i,j} \neq 0\}$ and $d' = d - c_0 - r_0$. Then*

$$(w_{c_0+i, r_0+j})_{(i,j) \in V_{d'}} \in \mathbb{Z}^{V_{d'}}$$

is a valid outcome. In particular, if $c_0 = r_0 = 0$, then w is a valid outcome.

Proof. We may assume that $d < \infty$. First we suppose that $c_0 = r_0 = 0$. Then the equations $\varphi_{0,d}(w) = 0$ and $\varphi_{d,0}(w) = 0$ show that $w_{0,j} < 0$ and $w_{i,0} < 0$ for some $i, j \in \{0, \dots, d\}$. Since $\#\text{supp}^-(w) = 1$, it follows that $i = j = 0$ and $\text{supp}^-(w) = \{(0, 0)\}$. Hence w is indeed valid.

In general, we note that φ_{c_0+a, r_0+b} vanishes on w for all $(a, b) \in V_{d'} \setminus V_{d'-1}$. So $\varphi_{a,b}$ vanishes on $(w_{c_0+i, r_0+j})_{(i,j) \in V_{d'}}$ for all $(a, b) \in V_{d'} \setminus V_{d'-1}$. This means that $(w_{c_0+i, r_0+j})_{(i,j) \in V_{d'}}$ is an outcome to which we can apply the previous case. □

Proposition 3.23. *Let $w = (w_{i,j})_{(i,j) \in V_d}$ be a valid outcome. If $w_{0,0} = 0$, then w is the initial configuration.*

Proof. This follows directly from Lemma 3.21. □

4. FROM REDUCED MODELS TO VALID OUTCOMES AND BACK

In this section, we continue to use the notion of a chipsplitting game and related concepts (Definitions 3.1 and 3.5). We augment this notion by allowing chip configurations to have rational or real entries (see Remark 3.2). We start by establishing a further characterization of the space of outcomes.

Lemma 4.1. *The space of integral (resp. rational, real) outcomes equals the kernel of the linear map*

$$\begin{aligned} \alpha_d: R^{V_d} &\rightarrow R[t]_{\leq d} \\ (w_{i,j})_{(i,j) \in V_d} &\mapsto \sum_{(i,j) \in V_d} w_{i,j} t^i (1-t)^j \end{aligned}$$

where $R = \mathbb{Z}$ (resp. $R = \mathbb{Q}, \mathbb{R}$).

Proof. The map α_∞ is the direct limit of the maps α_e for $e < \infty$. So we may assume that $d < \infty$. In this case, we know that the space of outcomes has codimension $d + 1$ by Proposition 3.14(b). For a given polynomial $p = \sum_{j=0}^d c_j t^j \in R[t]_{\leq d}$, set $w_{i,j} = c_i$ when $j = 0$ and $w_{i,j} = 0$ otherwise. Then $\alpha_d(w_{i,j})_{(i,j) \in V_d} = p$. So we see that α_d is surjective. Hence the kernel of α_d has the same codimension as the space of outcomes. It now suffices to show that every outcome is contained in the kernel of α_d . Note that the initial configuration is contained in the kernel of α_d . And, for $w \in R^{V_d}$, the value of $\alpha_d(w)$ does not change when we execute a chipsplitting move at $(i, j) \in V_{d-1}$. Indeed, we have

$$-t^i(1-t)^j + t^{i+1}(1-t)^j + t^i(1-t)^{j+1} = t^i(1-t)^j(-1+t+(1-t)) = 0$$

and so every outcome is contained in the kernel of α_d . \square

Let $\mathcal{M} = (w_\nu, i_\nu, j_\nu)_{\nu=0}^n$ be a reduced model. Then this model induces a real chip configuration $w(\mathcal{M}) = (w_{i,j})_{(i,j) \in V_\infty}$ by setting

$$w_{i,j} := \begin{cases} -1 & \text{if } (i, j) = (0, 0), \\ w_\nu & \text{if } (i, j) = (i_\nu, j_\nu), \\ 0 & \text{otherwise} \end{cases}$$

We have the following result.

Proposition 4.2.

- (a) *The map $\mathcal{M} \mapsto w(\mathcal{M})$ is a bijection between the set of reduced models and the set of valid real outcomes $w \in \mathbb{R}^{V_\infty}$ with $w_{0,0} = -1$.*
- (b) *Let S be the support of \mathcal{M} . Then $\text{supp}^+(w(\mathcal{M})) = S$.*
- (c) *The map $\mathcal{M} \mapsto w(\mathcal{M})$ is degree-preserving.*
- (d) *The chip configuration $w(\mathcal{M})$ is rational if and only if the coefficients of \mathcal{M} are all rational.*
- (e) *Every valid rational outcome $w \in \mathbb{Q}^{V_\infty}$ is of the form $\lambda \hat{w}$ for some $\lambda \in \mathbb{Q}_{>0}$ and valid integral outcome $\hat{w} \in \mathbb{Z}^{V_\infty}$.*
- (f) *Let $w \in \mathbb{R}^{V_\infty}$ be a valid real outcome with $w_{0,0} = 0$. Then $w = 0$.*

Proof. (a) From Lemma 4.1, it follows that $w(\mathcal{M})$ is indeed a valid real outcome with value -1 at $(0, 0)$. Clearly, the map $\mathcal{M} \mapsto w(\mathcal{M})$ is injective. Let $w \in \mathbb{R}^{V_\infty}$ be a valid real outcome with $w_{0,0} = -1$ and write $\text{supp}^+(w) = \{(i_0, j_0), \dots, (i_n, j_n)\}$ and take $w_\nu := w_{i_\nu, j_\nu}$ for $\nu = 0, \dots, n$. Then $(w_\nu, i_\nu, j_\nu)_{\nu=0}^n$ is a reduced model by Lemma 4.1. Hence the map $\mathcal{M} \mapsto w(\mathcal{M})$ is also surjective.

(b)–(d) hold by definition.

(e) For every valid rational outcome $w \in \mathbb{Q}^{V_\infty}$ there exist an $n \in \mathbb{N}$ such that $nw_{i,j} \in \mathbb{Z}$ for all (i, j) in the finite set $\text{supp}(w)$. Take $\hat{w} := nw$ and $\lambda := 1/n \in \mathbb{Q}_{>0}$. Then $\hat{w} \in \mathbb{Z}^{V_\infty}$ is a valid integral outcome using Lemma 4.1 and $w = \lambda \hat{w}$.

(f) Since w is an outcome with $\text{supp}^-(w) = \emptyset$, we know by Lemma 4.1 that $\sum_{(i,j)} w_{i,j} t^i (1-t)^j = 0$ for (i, j) ranging over $\text{supp}^+(w)$ and, by evaluating at $t = 1/2$, we see that $\text{supp}^+(w)$ can only be the empty set. Hence $w = 0$. \square

Proposition 4.3. *Theorems 1.1 and 3.6 are equivalent.*

Proof. By Remark 2.10, we know that for Theorem 1.1 it suffices to only consider fundamental models. Since the constraint $\sum_\nu p_\nu = 1$ of Definition 2.6 has coefficients in the rational numbers, the coefficients of a fundamental model are rational. Hence it suffices to only consider rational coefficients.

By Proposition 4.2 (e), every valid rational outcome is a positive multiple of a valid integral outcome. The space of outcomes is closed under scaling, and scaling does not change the degree or size of the positive support of a chip configuration. Hence for Theorem 3.6 it suffices to consider all valid rational outcomes w with $w_{0,0} = -1$.

The required equivalence is now given by Proposition 4.2 (a)–(d). \square

Next, we consider the chipsplitting equivalent of fundamental models.

Definition 4.4. A valid outcome $w \in \mathbb{Z}^{V_d} \setminus \{0\}$ is called *fundamental* if it cannot be written as

$$w = \mu_1 w_1 + \mu_2 w_2,$$

where $\mu_1, \mu_2 \in \mathbb{Q}_{>0}$ and $w_1, w_2 \in \mathbb{Z}^{V_d}$ are valid outcomes with $\text{supp}^+(w_1), \text{supp}^+(w_2) \subsetneq \text{supp}^+(w)$.

Applying Proposition 2.9 and keeping track of rational coefficients, we conclude the following.

Proposition 4.5. *Let \mathcal{M} be a reduced model with rational coefficients and let $n \in \mathbb{N}$ be any integer such that $w = nw(\mathcal{M})$ is an integral chip configuration. Then \mathcal{M} is a fundamental model if and only if w is a fundamental outcome.* \square

Thus we see that fundamental models correspond one-to-one with fundamental integral outcomes w with $\gcd\{w_{i,j} \mid (i,j) \in \text{supp}(w)\} = 1$.

5. VALID OUTCOMES OF POSITIVE SUPPORT ≤ 3

From now on, we will always assume that $d < \infty$. Since every chip configuration has finite degree, this assumption is harmless. In this section, we prove Theorem 3.6 for valid outcomes whose positive support has size ≤ 3 . To do this, we introduce our first tool, the Invertibility Criterion, which shows that certain subsets of V_d cannot contain the support of an outcome.

5.1. The Invertibility Criterion. Let $S \subseteq V_d$ and $E \subseteq \{0, \dots, d\}$ be nonempty subsets of the same size $\leq d + 1$. We start with the following definition.

Definition 5.1. We define

$$A_{E,S}^{(d)} := \left(\binom{d - \deg(i,j)}{a - i} \right)_{a \in E, (i,j) \in S}$$

to be the *pairing matrix* of (E, S) .

Let $w = (w_{i,j})_{(i,j) \in V_d} \in \mathbb{Z}^{V_d}$ be an outcome such that $\text{supp}(w) \subseteq S$.

Proposition 5.2 (Invertibility Criterion). *If $A_{E,S}^{(d)}$ is invertible, then w is the initial configuration.*

Proof. Suppose that $\text{supp}(w) \neq \emptyset$. Then

$$(w_{i,j})_{(i,j) \in S} \neq 0, \quad A_{E,S}^{(d)} \cdot (w_{i,j})_{(i,j) \in S} = (\varphi_{a,d-a}(w))_{a \in E} = 0$$

and hence $A_{E,S}^{(d)}$ is degenerate. \square

Our goal is to construct, for many subsets $S \subseteq V_d$, a subset E such that $A_{E,S}^{(d)}$ is invertible. We do this by dividing the pairing matrix into small parts and dealing with these parts separately.

5.2. **Divide.** Let $\lambda = (\lambda_1, \dots, \lambda_\ell) \in \mathbb{N}^\ell$ be a tuple of integers adding up to $d+1$. Write $c_i = \lambda_1 + \dots + \lambda_i$ for $i \in \{0, \dots, \ell\}$. For $k \in \{1, \dots, \ell\}$, let $S_k := \{(i, j) \in S \mid c_{k-1} \leq i < c_k\}$. Assume that the condition

$$\#S_k \in \{0, \lambda_k\}$$

is satisfied for every $k \in \{1, \dots, \ell\}$. Lastly, set

$$E_k := \begin{cases} \{c_{k-1}, c_{k-1} + 1, \dots, c_k - 1\} & \text{if } \#S_k = \lambda_k, \\ \emptyset & \text{if } S_k = \emptyset, \end{cases}$$

where the top row indicates consecutive integers ranging from c_{k-1} to $c_k - 1$.

Remark 5.3. Not all tuples λ will satisfy the condition that $\#S_k \in \{0, \lambda_k\}$ for all k . One can try to define a λ with this property recursively by, for $k = 1, 2, \dots$, picking λ_k minimal such that $\#S_k \in \{0, \lambda_k\}$. We stop when $c_k = d + 1$. This will work exactly when

$$\#\{(i, j) \in S \mid i \geq d - k\} \leq k + 1$$

for all $k \in \{0, 1, \dots, d\}$.

Proposition 5.4. Take $E = E_1 \cup \dots \cup E_\ell$. Then $\#E = \#S$ and we have

$$A_{E,S}^{(d)} = \begin{pmatrix} A_{E_1, S_1}^{(d)} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ A_{E_\ell, S_1}^{(d)} & \cdots & \cdots & A_{E_\ell, S_\ell}^{(d)} \end{pmatrix}.$$

In particular, the matrix $A_{E,S}^{(d)}$ is invertible if and only if all of $A_{E_1, S_1}^{(d)}, \dots, A_{E_\ell, S_\ell}^{(d)}$ are.

Proof. It is clear that $\#E = \#S$ and

$$A_{E,S}^{(d)} = \begin{pmatrix} A_{E_1, S_1}^{(d)} & \cdots & \cdots & A_{E_1, S_\ell}^{(d)} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ A_{E_\ell, S_1}^{(d)} & \cdots & \cdots & A_{E_\ell, S_\ell}^{(d)} \end{pmatrix}.$$

We need to show that $A_{E_k, S_{k'}}^{(d)} = 0$ when $k < k'$. Indeed, when $k < k'$, $a \in E_k$ and $(i, j) \in S_{k'}$, then

$$\binom{d - \deg(i, j)}{a - i} = 0$$

since $a < c_k \leq c_{k'-1} \leq i$. So $A_{E_k, S_{k'}}^{(d)} = 0$ when $k < k'$. \square

Example 5.5. Take $d = 6$ and let S be the set of positions marked with an $*$ below.

$$\begin{array}{ccccccc} \cdot & & & & & & \\ \cdot & \cdot & & & & & \\ * & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & * & \cdot & \\ * & \cdot & * & \cdot & \cdot & * & * \end{array}$$

The construction from Remark 5.3 yields the tuple $\lambda = (2, 1, 1, 1, 1, 1)$. We get

$$\begin{array}{ll} S_1 = \{(0, 0), (0, 4)\}, & E_1 = \{(0, 6), (1, 5)\}, \\ S_2 = \{(2, 0)\}, & E_2 = \{(2, 4)\}, \\ S_3 = \emptyset, & E_3 = \emptyset, \\ S_4 = \{(4, 1)\}, & E_4 = \{(4, 2)\}, \\ S_5 = \{(5, 0)\}, & E_5 = \{(5, 1)\}, \\ S_6 = \{(6, 0)\}, & E_6 = \{(6, 0)\}. \end{array}$$

So λ indeed satisfies the assumption and we see that

$$A_{E,S}^{(d)} = \begin{pmatrix} A_{E_1,S_1}^{(d)} & 0 & 0 & 0 & 0 \\ * & A_{E_2,S_2}^{(d)} & 0 & 0 & 0 \\ * & * & A_{E_4,S_4}^{(d)} & 0 & 0 \\ * & * & * & A_{E_5,S_5}^{(d)} & 0 \\ * & * & * & * & A_{E_6,S_6}^{(d)} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 & 0 & 0 \\ * & * & 1 & 0 & 0 & 0 \\ * & * & * & 1 & 0 & 0 \\ * & * & * & * & 1 & 0 \\ * & * & * & * & * & 1 \end{pmatrix}$$

is invertible. Hence S does not contain the support of a nonzero outcome.

5.3. **Conquer.** Using Proposition 5.4, we obtain matrices $A_{E,S}^{(d)}$ where

$$x := \min(E \cup \{i \mid (i, j) \in S\})$$

is relatively large.

Proposition 5.6. *Let $S' = \{(i - x, j) \mid (i, j) \in S\}$ and $E' = \{a - x \mid a \in E\}$. Then $A_{E,S}^{(d)} = A_{E',S'}^{(d-x)}$.*

Proof. This follows directly from the definition of the pairing matrix. \square

We now consider the case where $S \subseteq \{(i, j) \in V_d \mid i < j\}$ has s elements and $E = \{0, 1, \dots, s - 1\}$.

Proposition 5.7. *Suppose that one of the following holds:*

- (a) *We have $S = \{(0, i)\}$ for some $0 \leq i \leq d$ and $E = \{0\}$.*
- (b) *We have $S = \{(0, i), (0, j)\}$ for some $0 \leq i < j \leq d$ and $E = \{0, 1\}$.*
- (c) *We have $S = \{(0, i), (0, j), (0, k)\}$ for some $0 \leq i < j < k \leq d$ and $E = \{0, 1, 2\}$.*
- (d) *We have $S = \{(0, i), (0, j), (1, k)\}$ for some $0 \leq i < j \leq d$ and $0 \leq k \leq d - 1$ such that $i + j \neq 2k + 1$ and $E = \{0, 1, 2\}$.*

Then $A_{E,S}^{(d)}$ is invertible.

Proof. We prove the proposition case-by-case.

- (a) When $S = \{(0, i)\}$ for some $0 \leq i \leq d$ and $E = \{0\}$, we see that $A_{E,S}^{(d)} = (1)$ is invertible.
- (b) When $S = \{(0, i), (0, j)\}$ for some $0 \leq i < j \leq d$ and $E = \{0, 1\}$, we see that

$$A_{E,S}^{(d)} = \begin{pmatrix} 1 & 1 \\ d - i & d - j \end{pmatrix}$$

is invertible.

- (c) When $S = \{(0, i), (0, j), (0, k)\}$ for some $0 \leq i < j < k \leq d$ and $E = \{0, 1, 2\}$, we see that

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} A_{E,S}^{(d)} = \begin{pmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{pmatrix}$$

is a Vandermonde matrix, where $(x, y, z) = (d - i, d - j, d - k)$. Hence $A_{E,S}^{(d)}$ is invertible.

- (d) When $S = \{(0, i), (0, j), (1, k)\}$ for some $0 \leq i < j \leq d$ and $0 \leq k \leq d - 1$, we see that

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} A_{E,S}^{(d)} \begin{pmatrix} 1 & 1 \\ & -1 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & (x - y)^{-1} & \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ x^2 & x + y & 2z + 1 \end{pmatrix},$$

where $(x, y, z) = (d - i, d - j, d - 1 - k)$. Assume that $i + j \neq 2k + 1$. Then $x + y \neq 2z + 1$ and hence $A_{E,S}^{(d)}$ is invertible. \square

5.4. Valid outcomes of positive support ≤ 3 . We now classify the valid outcomes of positive support ≤ 3 . We start with the following lemma.

Lemma 5.8. *Let w be a valid outcome of degree $d \geq 1$. Then the following hold:*

- (a) *There are $i, j \in \{1, \dots, d\}$ such that $(0, 0), (i, 0), (0, j) \in \text{supp}^+(w)$.*
- (b) *There are distinct $i, j \in \{0, \dots, d\}$ such that $(i, d-i), (j, d-j) \in \text{supp}^+(w)$.*

Proof. (a) Since $\deg(w) > 0$, we see that w is not the initial configuration. Since w is valid, we therefore have $(0, 0) \in \text{supp}^-(w)$. Using $\psi_0(w) = \bar{\psi}_0(w) = 0$, we see that there are $i, j \in \{1, \dots, d\}$ such that $(i, 0), (0, j) \in \text{supp}^+(w)$.

(b) Since $\deg(w) = d$, there is an $i \in \{0, \dots, d\}$ such that $(i, d-i) \in \text{supp}^+(w)$. Using $\psi_d(w) = 0$, we see that there must also be a $j \in \{0, \dots, d\} \setminus \{i\}$ such that $(j, d-j) \in \text{supp}^+(w)$. \square

Proposition 5.9. *Let w be a valid degree- d outcome and assume that $\#\text{supp}^+(w) \leq 2$. Then*

$$\text{supp}^+(w) = \{(1, 0), (0, 1)\}.$$

Proof. By the previous lemma, we see that

$$\text{supp}(w) = \{(0, 0), (0, d), (d, 0)\} =: S.$$

Assume that $d \geq 2$. Then the construction from Remark 5.3 yields $\lambda = (2, 1, \dots, 1) \in \mathbb{N}^d$. We get $S_1 = \{(0, 0), (0, d)\}$, $S_k = \emptyset$ for $k \in \{2, \dots, d-1\}$ and $S_d = \{(d, 0)\}$. Using Propositions 5.4 and 5.6, we get

$$A_{\{0,1,d\},S}^{(d)} = \begin{pmatrix} A_{\{0,1\},S_1}^{(d)} & 0 \\ * & A_{\{d\},S_d}^{(d)} \end{pmatrix} = \begin{pmatrix} A_{\{0,1\},S_1}^{(d)} & 0 \\ * & A_{\{0\},\{(0,0)\}}^{(0)} \end{pmatrix}$$

and by Proposition 5.7 the submatrices on the diagonal are both invertible. So $A_{\{0,1,d\},S}^{(d)}$ is invertible. This contradicts the assumption that $\text{supp}(w) = S$ and so $d = 1$. \square

Lemma 5.10. *Let w be a valid degree- d outcome and assume that $\#\text{supp}^+(w) = 3$. Then one of the following holds:*

- (a) *We have $\text{supp}(w) = \{(0, 0), (d, 0), (0, d), (i, j)\}$ for some $i, j > 0$ with $\deg(i, j) < d$.*
- (b) *We have $\text{supp}(\sigma \cdot w) = \{(0, 0), (d, 0), (0, d), (e, 0)\}$ for some $\sigma \in S_3$ and $0 < e < d$.*
- (c) *We have $\text{supp}(\sigma \cdot w) = \{(0, 0), (d, 0), (0, e), (d-f, f)\}$ for some $\sigma \in S_2$ and $0 < e, f < d$.*

Proof. When $(d, 0), (0, d) \in \text{supp}(w)$, then it is easy to see that (a) or (b) holds. So suppose this is not the case. Since $\#\text{supp}^+(w) = 3$, we must have $(d, 0) \in \text{supp}(w)$ or $(0, d) \in \text{supp}(w)$ by Lemma 5.8. So there exists an $\sigma \in S_2$ such that $(d, 0) \in \text{supp}(\sigma \cdot w)$ and $(0, d) \notin \text{supp}(\sigma \cdot w)$. Now $\text{supp}(\sigma \cdot w) = \{(0, 0), (d, 0), (0, e), (d-f, f)\}$ for some $0 < e, f < d$ by Lemma 5.8. \square

We now apply the the Invertibility Criterion to the possible outcomes in each of these cases.

Proposition 5.11. *Let w be a degree- d outcome and assume that*

$$\text{supp}(w) = \{(0, 0), (d, 0), (0, d), (i, j)\}$$

for some $i, j > 0$ with $\deg(i, j) < d$. Then $d = 3$ and $(i, j) = (1, 1)$.

Proof. Assume that $i > 1$. Then the Invertibility Criterion combined with Propositions 5.4, 5.6 and 5.7 with $\lambda = (2, 1, \dots, 1)$ yields a contradiction. Indeed, we would find that

$$A_{\{0,1,i,d\},S}^{(d)} = \begin{pmatrix} A_{\{0,1\},S_1}^{(d)} & 0 & 0 \\ * & A_{\{i\},S_2}^{(d)} & 0 \\ * & * & A_{\{d\},S_3}^{(d)} \end{pmatrix}$$

is invertible where $S = S_1 \cup S_2 \cup S_3 = \{(0, 0), (0, d)\} \cup \{(i, j)\} \cup \{(d, 0)\}$. So $i = 1$. Applying the same argument to $(12) \cdot w$ shows that $j = 1$. Assume that $d > 3$. Then we apply the same strategy again with $\lambda = (3, 1, \dots, 1)$. We get a contradiction since

$$A_{\{0,1,2,d\},S}^{(d)} = \begin{pmatrix} A_{\{0,1,2\},S_1}^{(d)} & 0 \\ * & A_{\{d\},S_2}^{(d)} \end{pmatrix}$$

is invertible, where $S = S_1 \cup S_2 = \{(0, 0), (0, d), (1, 1)\} \cup \{(d, 0)\}$, by Proposition 5.7. So $d = 3$. \square

Proposition 5.12. *Let w be a degree- d outcome and assume that*

$$\text{supp}(w) = \{(0, 0), (d, 0), (0, d), (e, 0)\}$$

for some $0 < e < d$. Then $d = 2$ and $e = 1$.

Proof. The Invertibility Criterion with $\lambda = (2, 1, \dots, 1)$ yields $e = 1$. The Invertibility Criterion with $\lambda = (3, 1, \dots, 1)$ applied to $(12) \cdot w$ now yields $d = 2$. \square

Proposition 5.13. *Let w be a degree- d outcome and assume that*

$$\text{supp}(w) = \{(0, 0), (d, 0), (0, e), (d - f, f)\}$$

for some $0 < e, f < d$. Then $d = 2$ and $e = f = 1$.

Proof. The Invertibility Criterion with $\lambda = (2, 1, \dots, 1)$ yields $(d - f, f) = (1, d - 1)$. In particular, we have $e \leq f$. Applying the same argument to $(12) \cdot w$ with $\lambda = (2, 1, \dots, 1)$ if $e \neq f$ or $\lambda = (2, 1, \dots, 1, 2, 1, \dots, 1)$ if $e = f$, we find that $e = 1$. In the latter case, we have $E = \{0, 1, e, e + 1\}$ and $S = \{(0, 0), (0, d), (e, 0), (e, 1)\}$ so that

$$A_{E,S}^{(d)} = \begin{pmatrix} A_{\{0,1\},S_1}^{(d)} & 0 \\ * & A_{\{0,1\},S_2}^{(1)} \end{pmatrix}$$

where $S_1 = \{(0, 0), (0, d)\}$ and $S_2 = \{(0, 0), (0, 1)\}$. The Invertibility Criterion with $\lambda = (3, 1, \dots, 1)$ now yields $d = 2$. \square

Theorem 5.14. *Let w be a valid outcome of positive support ≤ 3 . Then w is a nonnegative multiple of one of the following outcomes:*

$$\begin{array}{cccccc} & & 1 & & & \\ & & \cdot & \cdot & 1 & \cdot & 1 \\ 1 & & \cdot & 3 & \cdot & \cdot & 2 & 1 & 1 & \cdot & 1 \\ -1 & 1 & -1 & \cdot & \cdot & 1 & -1 & \cdot & 1 & -1 & 1 & \cdot \end{array}$$

Proof. We know by the previous results that $\text{supp}^+(w)$ is one of the following:

$$\begin{aligned} & \{(0, 1), (1, 0)\}, \{(0, 3), (1, 1), (3, 0)\}, \{(0, 1), (0, 2), (2, 0)\}, \{(0, 2), (1, 0), (2, 0)\}, \\ & \{(0, 2), (1, 1), (2, 0)\}, \{(0, 1), (1, 1), (2, 0)\}, \{(0, 2), (1, 0), (1, 1)\}. \end{aligned}$$

For each of these possible supports E , we compute the space of outcomes whose supports are contained in $E \cup \{(0, 0)\}$ by computing the space of solutions to the Pascal equations of the corresponding degree. For each E , this space has dimension 1 (over \mathbb{Q}). We find that the outcomes with support

$$\{(0, 0), (0, 1), (0, 2), (2, 0)\} \text{ and } \{(0, 0), (0, 2), (1, 0), (2, 0)\}$$

are never valid. In each of the other cases, every valid outcome is a multiple of one in the list. \square

6. VALID OUTCOMES OF POSITIVE SUPPORT 4

In this section we prove Theorem 3.6 for valid outcomes whose positive support has size 4. To do this we introduce our second tool, the Hyperfield Criterion, which shows that certain subsets of V_d cannot be the support of a valid outcome. We first recall the basic properties of hyperfields.

6.1. **Polynomials over hyperfields.** Denote by 2^H the power set of a set H .

Definition 6.1. A *hyperfield* is a tuple $(H, +, \cdot, 0, 1)$ consisting of a set H , symmetric maps

$$- + -: H \times H \rightarrow 2^H \setminus \{\emptyset\}, \quad - \cdot -: H \times H \rightarrow H$$

and distinct elements $0, 1 \in H$ satisfying the following conditions:

- (a) The tuple $(H \setminus \{0\}, \cdot, 1)$ is a group.
- (b) We have $0 \cdot x = 0$ and $0 + x = \{x\}$ for all $x \in H$.
- (c) We have $\bigcup_{w \in x+y} (w + z) = \bigcup_{w \in y+z} (x + w)$ for all $x, y, z \in H$.
- (d) We have $a \cdot (x + y) = (a \cdot x) + (a \cdot y)$ for all $a, x, y \in H$.
- (e) For every $x \in H$ there is a unique element $-x \in H$ such that $x + (-x) \ni 0$.

For subsets $X, Y \subseteq H$, we write

$$X + Y := \bigcup_{x \in X, y \in Y} (x + y).$$

We also identify elements $y \in H$ with the singletons $\{y\} \subseteq H$ so that

$$y + X := X + y := \bigcup_{x \in X} (x + y).$$

With this notation, condition (c) can be reformulated as $(x + y) + z = x + (y + z)$ for all $x, y, z \in H$.

See [1] for more background and uses of hyperfields.

Definition 6.2. Let H be a hyperfield.

- (a) A *polynomial* in variables x_1, \dots, x_n over H is a formal sum

$$f = \sum_{k_1, \dots, k_n \in \mathbb{Z}_{\geq 0}} s_{k_1 \dots k_n} x_1^{k_1} \cdots x_n^{k_n}$$

such that $\#\{(k_1, \dots, k_n) \in \mathbb{Z}_{\geq 0}^n \mid s_{k_1 \dots k_n} \neq 0\} < \infty$.

- (b) We denote the set of such polynomials by $H[x_1, \dots, x_n]$.
- (c) For $s_1, \dots, s_n \in H$, we write

$$f(s_1, \dots, s_n) := \sum_{k_1, \dots, k_n \in \mathbb{Z}_{\geq 0}} s_{k_1 \dots k_n} s_1^{k_1} \cdots s_n^{k_n} \subseteq H.$$

and we say that f *vanishes* at (s_1, \dots, s_k) when $f(s_1, \dots, s_k) \ni 0$.

6.2. **The sign hyperfield.** For the remainder of this paper we let H be the *sign hyperfield*: it consists of the set $H = \{1, 0, -1\}$ with the addition defined by

$$0 + x = x, \quad 1 + 1 = 1, \quad (-1) + (-1) = -1, \quad 1 + (-1) = \{1, 0, -1\}$$

and the usual multiplication.

Definition 6.3. Let H be the sign hyperfield and let

$$f = \sum_{k_1, \dots, k_n \in \mathbb{Z}_{\geq 0}} c_{k_1 \dots k_n} x_1^{k_1} \cdots x_n^{k_n} \in \mathbb{R}[x_1, \dots, x_n]$$

be a polynomial. Then we call

$$\text{sign}(f) := \sum_{k_1, \dots, k_n \in \mathbb{Z}_{\geq 0}} \text{sign}(c_{k_1 \dots k_n}) x_1^{k_1} \cdots x_n^{k_n} \in H[x_1, \dots, x_n]$$

the polynomial over H *induced* by f . We also write

$$\text{sign}(w) := (\text{sign}(w_1), \dots, \text{sign}(w_n))$$

for all $w = (w_1, \dots, w_n) \in \mathbb{R}^n$.

Let ϕ be a Pascal equation on \mathbb{Z}^{V_d} . Then we can represent $\text{sign}(\phi)$ as a triangle consisting of the symbols $+, \cdot, -$ indicating that a given coefficient equals $1, 0, -1$, respectively.

Example 6.4. Take $d = 5$. Then the linear forms $\text{sign}(\varphi_{k,d-k})$ for $k = 0, \dots, d$ can be depicted as:

```

+           .           .           .           .           .
+ .         + +       . .         . .         . .         . .
+ . .       + + .     + + +     . . .     . . .     . . .
+ . . .     + + . .   + + + .   + + + +   . . . .   . . . .
+ . . . .   + + . . . + + + . . + + + + . + + + + +
+ . . . . . + + . . . . + + + . . . + + + + + . + + + + +
    
```

The linear forms $\text{sign}(\psi_k)$ for $k = 0, \dots, d$ can be depicted as:

```

.           .           .           .           .           +
. .         . .         . .         . .         + +         . -
. . .       . . .       . . .       + + +       . - -       . . +
. . . .     . . . .     + + + +     . - - -     . . + +     . . . -
. . . . .   + + + + +   . - - - -   . . + + +   . . . - -   . . . . +
+ + + + + + . - - - - - . . + + + + . . . - - - . . . . + + . . . . -
    
```

The linear forms $\text{sign}(\bar{\psi}_k)$ for $k = 0, \dots, d$ can be depicted as:

```

+           -           +           -           +           -
+ .         - +       + -       - +       + -       . +
+ . .       - + .     + - +     - + -     . - +     . . -
+ . . .     - + . .   + - + .   . + - +   . . + -   . . . +
+ . . . .   - + . . . . - + . . .   . . - + .   . . . - +   . . . . -
+ . . . . . . + . . . . . . + . . . . . . + . . . . + . . . . +
    
```

Proposition 6.5. Let H be the sign hyperfield and $f_1, \dots, f_k \in \mathbb{R}[x_1, \dots, x_n]$ polynomials. Suppose that f_1, \dots, f_k vanish at $w \in \mathbb{R}^n$. Then $\text{sign}(f_1), \dots, \text{sign}(f_k)$ vanish at $\text{sign}(w) \in H^n$.

Proof. Write $w = (w_1, \dots, w_n)$, $s = (s_1, \dots, s_n) = \text{sign}(w)$ and

$$f_i = \sum_{k_1, \dots, k_n \in \mathbb{Z}_{\geq 0}} c_{k_1 \dots k_n} x_1^{k_1} \dots x_n^{k_n}.$$

Then we have

$$\sum_{k_1, \dots, k_n \in \mathbb{Z}_{\geq 0}} c_{k_1 \dots k_n} w_1^{k_1} \dots w_n^{k_n} = f_i(w) = 0.$$

If $c_{k_1 \dots k_n} w_1^{k_1} \dots w_n^{k_n} = 0$ for all $k_1, \dots, k_n \in \mathbb{Z}_{\geq 0}$, then $\text{sign}(f_i)(s_1, \dots, s_n) = \{0\} \ni 0$ since all summands are zero. Otherwise, we have $c_{\ell_1 \dots \ell_n} w_1^{\ell_1} \dots w_n^{\ell_n} > 0$ for some $\ell_1, \dots, \ell_n \in \mathbb{Z}_{\geq 0}$ and $c_{\ell'_1 \dots \ell'_n} w_1^{\ell'_1} \dots w_n^{\ell'_n} < 0$ for some $\ell'_1, \dots, \ell'_n \in \mathbb{Z}_{\geq 0}$. In this case, $\text{sign}(f_i)(s_1, \dots, s_n)$ has both 1 and -1 as summands, so $\text{sign}(f_i)(s_1, \dots, s_n) = H \ni 0$. \square

6.3. The Hyperfield Criterion. We now state the Hyperfield Criterion. Let $S \subseteq V_d \setminus \{(0, 0)\}$ be a subset and define $s = (s_{i,j})_{(i,j) \in V_d} \in H^{V_d}$ by

$$s_{i,j} := \begin{cases} -1 & \text{when } (i,j) = (0,0), \\ 1 & \text{when } (i,j) \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Let $w = (w_{i,j})_{(i,j) \in V_d} \in \mathbb{Z}^{V_d}$ be a valid outcome.

Proposition 6.6 (Hyperfield Criterion). *Suppose that $\text{sign}(\phi)$ does not vanish at s for some Pascal equation ϕ on \mathbb{Z}^{V_d} . Then $\text{supp}^+(w) \neq S$.*

Proof. Suppose that $\text{supp}^+(w) = S$. Then $\text{sign}(w) = s$. Since all Pascal equations ϕ on \mathbb{Z}^{V_d} vanish at w , we see that all polynomials over H induced by Pascal equations on \mathbb{Z}^{V_d} vanish at s by Proposition 6.5. \square

6.4. Pascal equations. In this subsection, we consider the equations over H induced by the Pascal equations $\psi_k, \bar{\psi}_k, \varphi_{a,b}$ for $k \in \{0, \dots, d\}$ and $(a, b) \in V_d$ of degree d .

Definition 6.7. Let $s = (s_{i,j})_{(i,j) \in V_d} \in H^{V_d}$.

- (a) We call s a *sign configuration*.
- (b) The *positive support* of s is $\text{supp}^+(s) := \{(i, j) \in V_d \mid s_{i,j} = 1\}$.
- (c) The *negative support* of s is $\text{supp}^-(s) := \{(i, j) \in V_d \mid s_{i,j} = -1\}$.
- (d) The *support* of s is $\text{supp}(s) := \{(i, j) \in V_d \mid s_{i,j} \neq 0\} = \text{supp}^+(s) \cup \text{supp}^-(s)$.
- (e) We call $\text{deg}(s) := \max\{\text{deg}(i, j) \mid (i, j) \in V_d, s_{i,j} \neq 0\}$ the *degree* of s .
- (f) We say that s is *valid* when $s = 0$ or $\text{supp}^-(s) = \{(0, 0)\}$.
- (g) We say that s is *weakly valid* when for all $(i, j) \in \text{supp}^-(s)$ one of the following holds:
 - (a) $0 \leq i, j \leq 3$,
 - (b) $0 \leq i \leq 3$ and $\text{deg}(i, j) \geq d - 3$, or
 - (c) $0 \leq j \leq 3$ and $\text{deg}(i, j) \geq d - 3$.

Lemma 6.8. Let $w \in \mathbb{Z}^{V_d}$ be a chip configuration.

- (a) We have $\text{supp}^+(\text{sign}(w)) = \text{supp}^+(w)$.
- (b) We have $\text{supp}^-(\text{sign}(w)) = \text{supp}^-(w)$.
- (c) We have $\text{deg}(\text{sign}(w)) = \text{deg}(w)$.
- (d) The sign configuration $\text{sign}(w)$ is (weakly) valid if and only if w is (weakly) valid.

Proof. This follows from the definitions. □

Lemma 6.9.

- (a) We have

$$\text{sign}(\varphi_{a,b}) = \sum_{i=0}^a \sum_{j=0}^b x_{i,j}$$

for all $(a, b) \in V_d$ of degree d .

- (b) We have

$$\text{sign}(\psi_k) = \sum_{(i,j) \in S_k^+} x_{i,j} - \sum_{(i,j) \in S_k^-} x_{i,j} \text{ and } \text{sign}(\bar{\psi}_k) = \sum_{(i,j) \in S_k^+} x_{j,i} - \sum_{(i,j) \in S_k^-} x_{j,i},$$

where

$$\begin{aligned} S_k^+ &:= \{(i, j) \mid 0 \leq j \leq k, k - j \leq i \leq d - j, j \equiv k \pmod{2}\}, \\ S_k^- &:= \{(i, j) \mid 0 \leq j \leq k, k - j \leq i \leq d - j, j \not\equiv k \pmod{2}\}, \end{aligned}$$

for all $k \in \{0, \dots, d\}$.

Proof. This follows from Propositions 3.18 and 3.14. □

Proposition 6.10. Let $s \in H^{V_d}$ be a valid sign configuration of degree $d \geq 1$.

- (a) For $(a, b) \in V_d$ of degree d , if $\text{sign}(\varphi_{a,b})$ vanishes at s , then $\text{sign}(\varphi_{a,b})(s) = H$.
- (b) If $\text{sign}(\psi_0), \dots, \text{sign}(\psi_d)$ vanish at s , then $\text{sign}(\psi_0)(s) = \dots = \text{sign}(\psi_d)(s) = H$.
- (c) If $\text{sign}(\bar{\psi}_0), \dots, \text{sign}(\bar{\psi}_d)$ vanish at s , then $\text{sign}(\bar{\psi}_0)(s) = \dots = \text{sign}(\bar{\psi}_d)(s) = H$.

Proof. Note that since $\text{deg}(s) = d \geq 1$, we have $s_{0,0} = -1$, $s_{i,j} \geq 0$ for all $(i, j) \in V_d \setminus \{(0, 0)\}$ and $s_{k,d-k} = 1$ for some $k \in \{0, \dots, d\}$.

- (a) Let $(a, b) \in V_d$ have degree d and suppose that

$$\sum_{i=0}^a \sum_{j=0}^b s_{i,j} \ni 0.$$

Since $s_{0,0} = -1$, this is only possible when $s_{i,j} = 1$ for some $i \in \{0, \dots, a\}$ and $j \in \{0, \dots, b\}$ and so $\text{sign}(\varphi_{a,b})(s) = H$.

(b) Suppose that $\text{sign}(\psi_0), \dots, \text{sign}(\psi_d)$ vanish at s . We have

$$\text{sign}(\psi_k) = \sum_{(i,j) \in S_k^+} x_{i,j} - \sum_{(i,j) \in S_k^-} x_{i,j}$$

where $S_k^+, S_k^- \subseteq V_d$ are as in Lemma 6.9. We have $\psi_0 = \varphi_{d,0}$ and so $\text{sign}(\psi_0)(s) = \text{sign}(\varphi_{d,0})(s) = H$. For $k > 0$, note that $(0,0) \notin S_k^+ \cup S_k^-$ and in particular $s_{i,j} \geq 0$ for all $(i,j) \in S_k^+ \cup S_k^-$. So for each $k \in \{1, \dots, d\}$, we see that either

- (a_k) $s_{i,j} = 0$ for all $(i,j) \in S_k^+ \cup S_k^-$; or
- (b_k) $s_{i,j} = 1$ for some $(i,j) \in S_k^+$ and $s_{i,j} = 1$ for some $(i,j) \in S_k^-$.

We prove that (b_k) holds for $k = d, \dots, 1$ recursively, which implies that $\text{sign}(\psi_k)(s) = H$.

The union $S_d^+ \cup S_d^-$ consists of all points in V_d of degree d . So (a_d) cannot hold. So (b_d) holds. Next, let $k \in \{1, \dots, d-1\}$ and suppose that (b_{k+1}) holds. Then $s_{i,j} = 1$ for some $(i,j) \in S_{k+1}^-$. We have $S_{k+1}^- \subseteq S_k^+$ and hence (a_k) cannot hold. Hence (b_k) holds. So (b_k) holds for all $k \in \{1, \dots, d\}$.

(c) The proof of this part is the same as that of the previous part. \square

Remark 6.11. Let $w \in \mathbb{Z}^{V_d}$ be a valid outcome of degree d . Then $\text{sign}(\phi)$ vanishes at

$$s = (s_{i,j})_{(i,j) \in V_d} = \text{sign}(w)$$

for all Pascal equations ϕ on \mathbb{Z}^{V_d} . Proposition 6.10 tells us that in this case, we have

$$\text{sign}(\varphi_{0,d})(s), \dots, \text{sign}(\varphi_{d,0})(s), \text{sign}(\psi_0)(s), \dots, \text{sign}(\psi_d)(s), \text{sign}(\bar{\psi}_0)(s), \dots, \text{sign}(\bar{\psi}_d)(s) = H,$$

which shows that the following hold:

- (a) for all $(a,b) \in V_d$ of degree d , there exist $i \in \{0, \dots, a\}$ and $j \in \{0, \dots, b\}$ with $s_{i,j} = 1$;
- (b) for all $k \in \{1, \dots, d\}$, there exist $(i,j) \in S_k^+$ with $s_{i,j} = 1$ and $(i,j) \in S_k^-$ with $s_{i,j} = 1$; and
- (c) for all $k \in \{1, \dots, d\}$, there exist $(i,j) \in S_k^+$ with $s_{j,i} = 1$ and $(i,j) \in S_k^-$ with $s_{j,i} = 1$.

Here we note that $s_{i,j} = 1$ if and only if $(i,j) \in \text{supp}^+(w)$. So we can view these conditions as restrictions on the set $\text{supp}^+(w)$.

6.5. Contractions of hyperfield solutions. In this subsection, we make progress by considering the four-entries thick outer ring of the triangle V_d . We divide the outer ring into six areas as illustrated in Figure 5. One of these, Area D , splits further into $D^{(0)}$ and $D^{(1)}$ according to the parity of the i -coordinate of its entries.

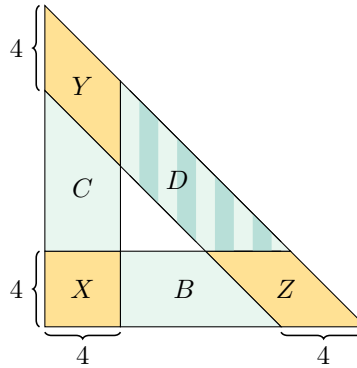


FIGURE 5. Dividing the outer ring of the triangle V_d into six areas for Subsection 6.5. The area D splits into two parts $D^{(0)}$ and $D^{(1)}$ by alternating the columns.

Then we see that

$$\begin{aligned}
\text{sign}(\psi_{d-3}) &= (-1)^{d-3} \sum_{j=0}^{d-3} \sum_{i=0}^3 (-1)^j x_{d-3-j+i,j} \\
&= \sum_{i=0}^3 \sum_{j=0}^i (-1)^{i+j} x_{i,d-3-i+j} - \sum_{i=0}^1 \sum_{k=0}^3 (-1)^{i+k} d_k^{(i)} + \sum_{j=0}^3 \sum_{i=0}^3 (-1)^{d-1+j} x_{d-3-j+i,j} \\
&= \begin{cases} \widehat{\psi}_{d-3}^{\text{even}} & \text{when } d \text{ is even,} \\ \widehat{\psi}_{d-3}^{\text{odd}} & \text{when } d \text{ is odd.} \end{cases}
\end{aligned}$$

Indeed, the linear forms $\widehat{\phi}^{\text{even}}, \widehat{\phi}^{\text{odd}}$ are the same for every $d \geq 12$.

Next we carry out the same subdivision as above but with the coordinates of the elements $s \in H^{V_d}$ instead of formal variables. We start by defining the index set

$$\Xi = \{0, 1, 2, 3\}^2 \sqcup \{0, 1, 2, 3\}^2 \sqcup \{0, 1, 2, 3\}^2 \sqcup \{0, 1, 2, 3\} \sqcup \{0, 1, 2, 3\} \sqcup \{0, 1, 2, 3\} \sqcup \{0, 1, 2, 3\}.$$

We write elements of H^Ξ as

$$\theta = (s, r, t, \alpha, \beta, \gamma) = \left((s_{i,j})_{i,j=0}^3, (r_{i,j})_{i,j=0}^3, (t_{i,j})_{i,j=0}^3, (\alpha_i)_{i=0}^3, (\beta_j)_{j=0}^3, (\gamma_k^{(0)})_{k=0}^3, (\gamma_k^{(1)})_{k=0}^3 \right).$$

Definition 6.16.

- (a) We say that θ is *valid* when $\theta = 0$ or when $s_{0,0} = -1$ and $r_{i,j}, t_{i,j}, \alpha_i, \beta_j, \gamma_k^{(0)}, \gamma_k^{(1)} \geq 0$ for all $i, j, k \in \{0, \dots, 3\}$ and $s_{i,j} \geq 0$ for all $(i, j) \in \{0, \dots, 3\}^2 \setminus \{(0, 0)\}$.
- (b) We say that θ is *weakly valid* when $\alpha_i, \beta_j, \gamma_k^{(0)}, \gamma_k^{(1)} \geq 0$ for all $i, j, k \in \{0, \dots, 3\}$.

Thus θ is weakly valid if and only if its negative support is contained in the areas X, Y, Z of Figure 5.

For $d \geq 11$ and $s = (s_{i,j})_{(i,j) \in V_d} \in H^{V_d}$ weakly valid, we write

$$\text{contr}_d(s) := \left((s_{i,j})_{i,j=0}^3, (r_{i,j})_{i,j=0}^3, (t_{i,j})_{i,j=0}^3, (\alpha_i)_{i=0}^3, (\beta_j)_{j=0}^3, (\gamma_k^{(0)})_{k=0}^3, (\gamma_k^{(1)})_{k=0}^3 \right) \in H^\Xi,$$

where we have

$$\begin{aligned}
r_{i,j} &:= s_{i,d-3-i+j} \quad \text{for } i, j \in \{0, \dots, 3\}, \\
t_{i,j} &:= s_{d-3-j+i,j} \quad \text{for } i, j \in \{0, \dots, 3\}, \\
\alpha_i &:= s_{i,4} + \dots + s_{i,d-4-i} \quad \text{for } i \in \{0, \dots, 3\}, \\
\beta_j &:= s_{4,j} + \dots + s_{d-4-j,j} \quad \text{for } j \in \{0, \dots, 3\}, \\
\gamma_k^{(0)} &:= \begin{cases} s_{4,d-4-k} + s_{6,d-6-k} + \dots + s_{d-4-k,4} & \text{when } d+k \equiv 0 \pmod{2} \\ s_{4,d-4-k} + s_{6,d-6-k} + \dots + s_{d-5-k,5} & \text{when } d+k \equiv 1 \pmod{2} \end{cases} \quad \text{for } k \in \{0, \dots, 3\}, \\
\gamma_k^{(1)} &:= \begin{cases} s_{5,d-5-k} + s_{7,d-7-k} + \dots + s_{d-4-k,4} & \text{when } d+k \equiv 1 \pmod{2} \\ s_{5,d-5-k} + s_{7,d-7-k} + \dots + s_{d-5-k,5} & \text{when } d+k \equiv 0 \pmod{2} \end{cases} \quad \text{for } k \in \{0, \dots, 3\}.
\end{aligned}$$

Let $s_1, \dots, s_k \in H \setminus \{-1\}$. Then $s_1 + \dots + s_k$ always consists of a single element, namely the element $\max(s_1, \dots, s_k)$. So the weakly valid assumption ensures that the hyperfield sums in this definition evaluate to a single element of H . Note that when $s \in H^{V_d}$ is (weakly) valid, then $\text{contr}_d(s)$ is (weakly) valid as well.

Let the coordinates $x_{i,j}, y_{i,j}, z_{i,j}, b_j, c_i, d_k^{(0)}, d_k^{(1)}$ be dual to $s_{i,j}, r_{i,j}, t_{i,j}, \alpha_i, \beta_j, \gamma_k^{(0)}, \gamma_k^{(1)}$. This allows us to view $\{\widehat{\phi} \mid \phi \in \Phi_1\}$, $\{\widehat{\phi}^{\text{even}} \mid \phi \in \Phi_2\}$ and $\{\widehat{\phi}^{\text{odd}} \mid \phi \in \Phi_2\}$ as sets of equations on H^Ξ . See Figure 6 for a visualisation of contr_d .

Definition 6.17. Let $\theta \in H^\Xi$. We define the *positive support* of θ to be the set $\text{supp}^+(\theta)$ of symbols $x_{i,j}, y_{i,j}, z_{i,j}, b_j, c_i, d_k^{(0)}, d_k^{(1)}$ with $i, j, k \in \{0, \dots, 3\}$ such that the symbol evaluated at θ equals 1.

Lemma 6.21. *Let $s \in H^{V_d}$ be valid of degree d such that $\#\text{supp}^+(s) \leq 4$.*

(a) *If $d = 6$, then $s \in \Omega_d$ if and only if $\text{supp}^+(s)$ is one of the following sets:*

$$\{(0, 3), (1, 5), (4, 1), (6, 0)\}, \{(0, 5), (1, 1), (3, 3), (6, 0)\}, \{(0, 6), (1, 1), (3, 3), (5, 0)\}, \\ \{(0, 6), (1, 1), (3, 3), (6, 0)\}, \{(0, 6), (1, 4), (3, 0), (5, 1)\}.$$

(b) *If $d = 7$, then $s \in \Omega_d$ if and only if $\text{supp}^+(s)$ is one of the following sets:*

$$\{(0, 7), (1, 1), (3, 3), (7, 0)\}, \{(0, 7), (1, 3), (5, 1), (7, 0)\}, \{(0, 7), (1, 5), (3, 1), (7, 0)\}.$$

(c) *If $d \in \{8, \dots, 11\}$, then $s \notin \Omega_d$.*

(d) *If $d \geq 12$, then $s \notin \Gamma_d$.*

Proof. Parts (a)-(c) are verified by computer. For (d), we verify by computer that Γ^{even} and Γ^{odd} do not contain any points whose positive support has size ≤ 4 . This is possible since the sets H^{Ξ} and Φ are finite. Thus by Proposition 6.20 we have $s \notin \Gamma_d$. \square

Proof of Theorem 6.20. Let $w \in \mathbb{Z}^{V_d}$ be a valid outcome and suppose that $\#\text{supp}^+(w) = 4$. We may assume that $\deg(w) = d$. Suppose that $\deg(w) \geq 6$. Take $s := \text{sign}(w)$. Then $s \in \Omega_d \subseteq \Gamma_d$. By Lemma 6.21, $\deg(s) \in \{6, 7\}$ and there are only 8 possibilities for $\text{supp}^+(w)$. In every case, it is easy to verify that no valid w with such a positive support exist using the Invertibility Criterion. Hence $\deg(w) \leq 5$. \square

7. VALID OUTCOMES OF POSITIVE SUPPORT 5

In this section we prove Conjecture 3.6 for valid outcomes whose positive support has size 5. To do this we introduce our third tool, the Hexagon Criterion, illustrated in Figure 7.

7.1. The Hexagon Criterion. Let $\ell_1, \ell_2 \geq d' \geq 1$ be integers such that $d' + \ell_1 + \ell_2 \leq d$. Let $w = (w_{i,j})_{(i,j) \in V_d} \in \mathbb{Z}^{V_d}$ be a chip configuration and write $w' = (w_{i,j})_{(i,j) \in V_{d'}} \in \mathbb{Z}^{V_{d'}}$.

Proposition 7.1 (Hexagon Criterion). *Suppose that*

$$\text{supp}(w) \subseteq V_{d'} \cup \{(i, j) \in V_d \mid j > d - \ell_1\} \cup \{(i, j) \in V_d \mid i > d - \ell_2\}$$

holds. Then the following statements hold:

(a) *If w' is not an outcome, then w is not an outcome.*

(b) *If w is a valid outcome, then $\deg(w) \leq d'$.*

Proof. (a) We suppose that w is an outcome and prove w' is also an outcome. For $k \in \{0, \dots, d'\}$, let $\widehat{\varphi}_k$ be the linear form obtained from $\varphi_{\ell_1+k, d-\ell_1-k}$ by setting $x_{i,j}$ to 0 for all $(i, j) \in V_d$ with $\deg(i, j) > d'$. Then $\widehat{\varphi}_0, \dots, \widehat{\varphi}_{d'}$ are Pascal equations on $\mathbb{Z}^{V_{d'}}$ and we have

$$\widehat{\varphi}_k(w') = \varphi_{\ell_1+k, d-\ell_1-k}(w) = 0$$

for all $k \in \{0, \dots, d'\}$. We next prove that these equations are linearly independent. For $a \in \{0, \dots, d'\}$ define $e^{(a)} = (e_{i,j}^{(a)})_{(i,j) \in V_{d'}} \in \mathbb{Z}^{V_{d'}}$ by

$$e_{i,j}^{(a)} := \begin{cases} 1 & \text{when } (i, j) = (a, d' - a), \\ 0 & \text{otherwise} \end{cases}$$

and consider the matrix

$$A = \left(\widehat{\varphi}_k(e^{(a)}) \right)_{k,a=0}^{d'} = \left(\binom{d-d'}{\ell_1+k-a} \right)_{k,a=0}^{d'} = \left(\binom{(d-d'-\ell_1)+\ell_1}{\ell_1+k-a} \right)_{k,a=0}^{d'}.$$

If A is invertible, then $\widehat{\varphi}_0, \dots, \widehat{\varphi}_{d'}$ must be linearly independent. Note that we have $0 \leq \ell_1 + k - a \leq d - d'$, so all entries of A are nonzero. Also note that $d - d' - \ell_1 \geq \ell_2 \geq 0$. Applying Theorem 8 in the note [3] with $a := d - d' - \ell_1, b := \ell_1$ and $c := d' + 1$ yields

$$\det(A) = \frac{H(\ell_1)H(d-d'-\ell_1)H(d'+1)H(d+1)}{H(d-d')H(d'+\ell_1+1)H(d-\ell_1+1)} \neq 0,$$

where $H(n) = 1!2! \cdots n!$. So A is invertible and $\widehat{\varphi}_0, \dots, \widehat{\varphi}_{d'}$ are $d' + 1$ linearly independent Pascal equations on $\mathbb{Z}^{V_{d'}}$. These equations must be a basis of the space of all Pascal equations on $\mathbb{Z}^{V_{d'}}$. Since $\widehat{\varphi}_0(w') = \dots = \widehat{\varphi}_{d'}(w') = 0$, it follows that w' is an outcome.

(b) Suppose that w is a valid outcome. Then w' must also be an outcome by part (a). Extend w' to an element $w'' \in \mathbb{Z}^{V_d}$ by setting $w''_{i,j} = w'_{i,j}$ for $(i,j) \in V_{d'}$ and $w''_{i,j} = 0$ for $(i,j) \in V_d$ with $\deg(i,j) > d'$. Then w'' is again an outcome. Now we see that $w - w''$ is an outcome with an empty negative support. So $w - w''$ must be the initial configuration by Lemma 3.21. Hence $w = w''$ has degree $\leq d'$. \square

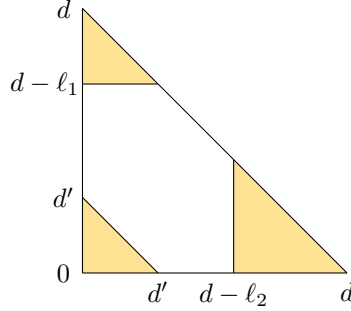


FIGURE 7. Illustration of the Hexagon Criterion. If w is an outcome whose support is contained in the orange area, then its restriction w' to the bottom-left orange triangle is also an outcome. If in addition w is valid, then $\text{supp}(w)$ is entirely contained in the bottom-left orange triangle.

7.2. Valid outcomes of positive support 5. We now use the Invertibility Criterion, Hyperfield Criterion and Hexagon Criterion to prove the following result.

Theorem 7.2. *Let $w \in \mathbb{Z}^{V_d}$ be a valid outcome and suppose that $\#\text{supp}^+(w) = 5$. Then $\deg(w) \leq 7$.*

Let $w \in \mathbb{Z}^{V_d}$ be a valid outcome and suppose that $\#\text{supp}^+(w) = 5$. We may assume that $\deg(w) = d$. To start, we verify by computer that $d \notin \{8, \dots, 41\}$ using the Hyperfield Criterion followed by the Invertibility Criterion. So we may assume that $d \geq 42$.

Our next step is to apply the Hyperfield Criterion as we did in the previous section. We have $\#\text{supp}^+(w) \leq 5$ and from this it follows that $\text{supp}^+(\text{contr}_d(\text{sign}(w)))$ also has size ≤ 5 . Recall that Γ^{even} and Γ^{odd} do not contain any elements with a positive support of size ≤ 4 . So $\text{contr}_d(\text{sign}(w)) \in \Gamma^{\text{even}} \cup \Gamma^{\text{odd}}$ must in fact have a positive support of size exactly 5. One can verify by computer that Γ^{even} contains 1283 elements whose positive support has size 5 and Γ^{odd} contains 1265 such elements. Basically, our strategy is to split into $1283 + 1265$ cases, and in each case assume that $\text{contr}_d(\text{sign}(w))$ is some fixed element of $\Gamma^{\text{even}} \cup \Gamma^{\text{odd}}$. If we can show that none of these cases can occur we are done.

Before doing this, we make one simplification: write

$$\Xi' := \{0, 1, 2, 3\}^2 \sqcup \{0, 1, 2, 3\}^2 \sqcup \{0, 1, 2, 3\}^2 \sqcup \{0, 1, 2, 3\} \sqcup \{0, 1, 2, 3\} \sqcup \{0, 1, 2, 3\}$$

and

$$\chi(s, r, t, \alpha, \beta, \gamma^{(0)}, \gamma^{(1)}) := (s, r, t, \alpha, \beta, \gamma^{(0)} + \gamma^{(1)})$$

for all weakly valid $(s, r, t, \alpha, \beta, \gamma^{(0)}, \gamma^{(1)}) \in H^{\Xi'}$, where the addition of $\gamma^{(0)}, \gamma^{(1)}$ is defined componentwise. The composition $\text{contr}'_d := \chi \circ \text{contr}_d$ can be visualized in the same way as contr_d . We again get Figure 6, but now $d_k^{(0)}$ and $d_k^{(1)}$ are replaced by d_k .

Let $\Lambda \subseteq H^{\Xi'}$ be the set of elements $\chi(\theta)$ with $\theta \in \Gamma^{\text{even}} \cup \Gamma^{\text{even}}$ of positive support 5. We will split into cases, where in each case the element $\text{contr}'_d(\text{sign}(w)) \in \Lambda$ is fixed.

Definition 7.3. Let $\theta' \in H^{\Xi'}$. We define the *positive support* of θ' to be the set $\text{supp}^+(\theta')$ of symbols $x_{i,j}, y_{i,j}, z_{i,j}, c_i, r_j, d_k$ with $i, j, k \in \{0, \dots, 3\}$ such that the symbol evaluated at θ equals 1.

Clearly, the elements of Λ have a positive support of size ≤ 5 . It turns out that the positive support actually has size 5 in all but one case.

Lemma 7.4. *Let $\theta' \in \Lambda$. Then exactly one of the following holds:*

- (a) *The element θ' has a positive support of size 5.*
- (b) *We have $\theta' = \chi(\theta)$ where $\theta \in H^{\Xi}$ is valid with $\text{supp}^+(\theta) = \{x_{0,3}, x_{1,1}, x_{3,0}, d_0^{(0)}, d_0^{(1)}\}$.*

Proof. This is verified by computer. □

We first deal with the second case.

Lemma 7.5. *Let $d \geq 12$. Then there is no weakly valid outcome $w = (w_{i,j})_{(i,j) \in V_d}$ such that*

$$\text{supp}^+(\text{contr}_d(\text{sign}(w))) = \{x_{0,3}, x_{1,1}, x_{3,0}, d_0^{(0)}, d_0^{(1)}\}.$$

Proof. Suppose that such an outcome w exists. Then we have

$$\text{supp}(w) = \{(0, 0), (0, 3), (1, 1), (3, 0), (i, d - i), (j, d - j)\}$$

for some $i, j \in \{4, \dots, d\}$ with i even and j odd. Let $u = (u_{i,j})_{(i,j) \in V_d}$ be the outcome with

$$\text{supp}(u) = \{(0, 0), (0, 3), (1, 1), (3, 0)\}$$

defined by $u_{0,0} = -1$, $u_{0,3} = u_{3,0} = 1$ and $u_{1,1} = 3$. Take $w' = w + w_{0,0}u \in \mathbb{Z}^{V_d}$. Note that w' is an outcome. We have

$$\{(i, d - i), (j, d - j)\} \subseteq \text{supp}(w') \subseteq \{(0, 3), (1, 1), (3, 0), (i, d - i), (j, d - j)\}.$$

We see that w' cannot be the initial configuration. On the other hand, the Invertibility Criterion with $\lambda = (1, \dots, 1)$ shows that w' must be the initial configuration. Contradiction. □

From now on, we assume that there exists a valid outcome $w \in \mathbb{Z}^{V_d}$ with $\#\text{supp}^+(w) = 5$ and $\deg(w) = d$ such that

$$\text{contr}'_d(\text{sign}(w)) = \theta'$$

for some fixed $\theta' \in \Lambda$ with a positive support of size 5. We have 2289 cases. Our goal is to prove that w cannot exist. We first have the following observation.

Lemma 7.6. *Let $\theta' \in \Lambda$ with a positive support of size 5.*

- (a) *The set $\text{supp}^+(\theta') \cap \{c_0, \dots, c_3\}$ has at most 1 element.*
- (b) *The set $\text{supp}^+(\theta') \cap \{r_0, \dots, r_3\}$ has at most 1 element.*
- (c) *The set $\text{supp}^+(\theta') \cap \{d_0, \dots, d_3\}$ has at most 1 element.*

Proof. This is verified by computer. □

Next, we will first extract information about w and put it into a form that the Invertibility Criterion can be applied to. We define the maps

$$\begin{aligned} \text{relcoord}: \{0, \dots, d\} &\rightarrow \{0, \dots, 3, M, d - 6, \dots, d\} \\ i &\mapsto \begin{cases} i & \text{when } i \in \{0, \dots, 3\}, \\ M & \text{when } i \in \{4, \dots, d - 7\}, \\ i & \text{when } i \in \{d - 6, \dots, d\} \end{cases} \end{aligned}$$

and

$$\begin{aligned} \text{relset}: \mathbb{Z}^{V_d} &\rightarrow 2^{\{0, \dots, 3, M, d - 6, \dots, d\}^2} \\ w &\mapsto \{(\text{relcoord}(i), \text{relcoord}(j)) \mid (i, j) \in \text{supp}(w)\} \end{aligned}$$

with M a new symbol and consider the possible sets $\text{relset}(w)$ given that $\text{contr}'_d(\text{sign}(w)) = \theta'$.

Lemma 7.7. *Write $\text{contr}'_d(\text{sign}(w)) = (s, r, t, \alpha, \beta, \gamma)$.*

- (a) For $i, j \in \{0, \dots, 3\}$, if $s_{i,j} \neq 0$, then $(i, j) \in \text{relset}(w)$.
- (b) For $i, j \in \{0, \dots, 3\}$, if $r_{i,j} \neq 0$, then $(i, d - 3 + j - i) \in \text{relset}(w)$.
- (c) For $i, j \in \{0, \dots, 3\}$, if $t_{i,j} \neq 0$, then $(d - 3 + i - j, j) \in \text{relset}(w)$.
- (d) For $i \in \{0, \dots, 3\}$, if $\alpha_i \neq 0$, then $\text{relset}(w) \cap \{(i, M), (i, d - 6), \dots, (i, d - 4 - i)\} \neq \emptyset$.
- (e) For $j \in \{0, \dots, 3\}$, if $\beta_j \neq 0$, then $\text{relset}(w) \cap \{(M, j), (d - 6, j), \dots, (d - 4 - j, j)\} \neq \emptyset$.
- (f) For $k \in \{0, \dots, 3\}$, if $\gamma_k \neq 0$, then

$$\text{relset}(w) \cap \{(M, d - 4 - k), \dots, (M, d - 6), (M, M), (d - 6, M), \dots, (d - 4 - k, M)\} \neq \emptyset.$$

Proof. Follows from the definition of relset . \square

We can use the Invertibility Criterion to prove that some subsets of $\{0, \dots, 3, M, d - 6, \dots, d\}^2$ are not of the form $\text{relset}(w)$ for an outcome $w \in \mathbb{Z}^{V_d}$ with $\#\text{relset}(w) = \#\text{supp}(w)$.

Example 7.8. Let $w \in \mathbb{Z}^{V_d}$ for $d \geq 12$. Suppose that $\#\text{supp}(w) = 7$ and

$$\text{relset}(w) = \{(0, 0), (0, d), (1, 3), (M, 2), (M, d - 6), (d - 5, M), (d, 0)\}.$$

We claim that w cannot be an outcome. Indeed, we have

$$\text{supp}(w) = \{(0, 0), (0, d), (1, 3), (i, 2), (j, d - 6), (d - 5, k), (d, 0)\}$$

for some $i, j, k \in \{4, \dots, d - 7\}$. We now partition $\text{supp}(w)$ as follows:

$$\begin{aligned} \text{supp}(w) &= \{(0, 0), (0, d), (1, 3)\} \cup \{(i, 2), (j, d - 6)\} \cup \{(d - 5, k)\} \cup \{(d, 0)\} \\ &= \{(0, 0), (0, d), (1, 3)\} \cup \{(i, 2)\} \cup \{(j, d - 6)\} \cup \{(d - 5, k)\} \cup \{(d, 0)\}. \end{aligned}$$

When $i = j$, we can apply the Invertibility Criterion with the first partition to see that no outcome with support $\text{supp}(w)$ exists. When $i \neq j$, we can apply the Invertibility Criterion with the second partition to get the same result. Hence w is not an outcome.

For using the Invertibility Criterion directly on subsets of $\{0, \dots, 3, M, d - 6, \dots, d\}^2$, we have the following observations.

- (a) We have at most two elements of the form (M, \bullet) . These elements originate from points $(i, \bullet) \in V_d$ with $4 \leq i \leq d - 7$. Assume that we have two such points (i, \bullet) and (i', \bullet) . Then we have to apply the Invertibility Criterion in a different way depending on whether i, i' are equal or not. We always assume the worst case, which is the case where $i = i'$. A similar statement holds for the at most two elements of the form (\bullet, M) .
- (b) Assume that we have elements $(i, x), (i, y), (i', z) \in V_d$ with $i < i'$ and $x < y$. Then we can apply the Invertibility Criterion as long as $x + y \neq 2z + 1$. In some cases, we can conclude that this condition holds when we only know $\text{relcoord}(x), \text{relcoord}(y), \text{relcoord}(z)$. For example, when $\text{relcoord}(x) \leq 3, \text{relcoord}(y) \geq d - 6, \text{relcoord}(z) \neq M$, then $x + y \neq 2z + 1$ since we assume that $d \geq 40$.

Given that $\text{contr}'_d(\text{sign}(w)) = \theta'$, we can now write down a finite list of possibilities for $\text{relset}(w)$. For each possibility, we attempt to show that w cannot exist using the Invertibility Criterion. When this is successful for all possibilities, we can discard the case $\text{contr}'_d(\text{sign}(w)) = \theta'$. In this way, we can reduce the number of possible cases to 1107. Next, we use symmetry to further reduce the number of cases. We have an action of S_3 of $H^{\Xi'}$ given by

$$(12) \cdot (s, r, t, \alpha, \beta, \gamma) := ((s_{j,i})_{i,j=0}^3, (t_{j,i})_{i,j=0}^3, (r_{j,i})_{i,j=0}^3, \beta, \alpha, \gamma),$$

$$(13) \cdot (s, r, t, \alpha, \beta, \gamma) := ((t_{3-i,j})_{i,j=0}^3, (r_{3-j,3-i})_{i,j=0}^3, (s_{3-i,j})_{i,j=0}^3, \gamma, \beta, \alpha)$$

for all $(s, r, t, \alpha, \beta, \gamma) \in H^{\Xi'}$. This action satisfies

$$\sigma \cdot \text{contr}'_d(\text{sign}(w)) = \text{contr}'_d(\sigma \cdot \text{sign}(w)) = \text{contr}'_d(\text{sign}(\sigma \cdot w))$$

for all weakly valid outcomes $w \in \mathbb{Z}^{V_d}$. This means that to exclude a particular case $\text{contr}'_d(\text{sign}(w)) = \theta'$, it suffices to prove that there are no weakly valid outcomes $w \in \mathbb{Z}^{V_d}$ with $\text{contr}'_d(\text{sign}(w)) = \sigma \cdot \theta'$ for some $\sigma \in S_3$. This allows us to reduce the number of possible cases further to 349.

Our last step is to apply the Hexagon Criterion to these 349 cases. First, assume that

$$(3) \quad \text{supp}^+(\theta') \cap \{c_0, \dots, c_3, r_0, \dots, r_3, d_0, \dots, d_3\} = \emptyset$$

holds. Then we can apply the Hexagon Criterion with $d' = 6$ and $\ell_1 = \ell_2 = 7$ since $d \geq 20$. We find that $20 \leq d = \deg(w) \leq d' = 6$. This is a contradiction and so each of the 325 cases satisfying (3) are not possible. This reduces the number of possible cases to 24.

Next, we assume that

$$(4) \quad \#\text{supp}^+(\theta') \cap \{c_0, c_1, r_0, r_1, d_0, d_1\} = 1 \text{ and } \#\text{supp}^+(\theta') \cap \{c_2, c_3, r_2, r_3, d_2, d_3\} = 0.$$

This means that

$$\text{supp}(w) \setminus \{(a, b)\} \subseteq V_6 \cup \{(i, j) \in V_d \mid j > d - 7\} \cup \{(i, j) \in V_d \mid i > d - 7\}$$

for some $(a, b) \in V_d$ with $a \leq 1$, $b \leq 1$ or $\deg(a, b) \geq d - 1$. Indeed, when $c_i \in \text{supp}^+(\theta')$ we get such an (a, b) with $a = i$, when $r_j \in \text{supp}^+(\theta')$ we get such an (a, b) with $b = j$ and when $d_k \in \text{supp}^+(\theta')$ we get such an (a, b) with $\deg(a, b) = d - k$. Now, at least one of the following holds:

- (a) We have $\deg(a, b) \leq \lfloor d/3 \rfloor$.
- (b) We have $a \geq \lfloor d/3 \rfloor$.
- (c) We have $b \geq \lfloor d/3 \rfloor$.

When $a \leq 1$ and $\deg(a, b) > \lfloor d/3 \rfloor$, we see that $b \geq \lfloor d/3 \rfloor$. When $b \leq 1$ and $\deg(a, b) > \lfloor d/3 \rfloor$, we see that $a \geq \lfloor d/3 \rfloor$. When $\deg(a, b) \geq d - 1$, then either $a \geq \lfloor d/3 \rfloor$ or $b \geq \lfloor d/3 \rfloor$. So indeed, one of these statements has to hold.

When (a) holds, then we can apply the Hexagon Criterion with $d' = \ell_1 = \ell_2 = \lfloor d/3 \rfloor \geq 7$ since $d \geq 21$. When (b) holds, then we use $d' = 6$, $\ell_1 = 7$ and $\ell_2 = d + 1 - \lfloor d/3 \rfloor$ instead. We can do this since $d \geq 42$. When (c) holds, then we use $d' = 6$, $\ell_1 = d + 1 - \lfloor d/3 \rfloor$ and $\ell_2 = 7$ instead. In each case, we find that $d = \deg(w) \leq d' < d$. This is a contradiction. Hence each of the 24 cases satisfying (4) are not possible.

This leaves one single case remaining where

$$\text{supp}^+(\theta') \cap \{c_0, \dots, c_3, r_0, \dots, r_3, d_0, \dots, d_3\}$$

consists of two elements. We deal with this case by hand.

Lemma 7.9. *There is no weakly valid outcome $w \in \mathbb{Z}^{V_d}$ such that*

$$\text{contr}'_d(\text{sign}(w)) = \{x_{0,0}, y_{0,3}, z_{3,0}, c_1, r_1, d_1\}.$$

Proof. Assume that such a w exists. The support of w is then of the form

$$S = \{(0, 0), (d, 0), (0, d), (i, 1), (1, j), (k, d - 1 - k)\}.$$

Write $d = 2e + 1$. When $j \neq e$, we see that S cannot be the support of an outcome using the Invertibility Criterion. Using symmetry, we similarly find that S cannot be the support of an outcome when $i \neq e$ or $k \neq e$. This leaves the case where

$$S = \{(0, 0), (d, 0), (0, d), (e, 1), (1, e), (e, e)\}.$$

Now we take $E = \{0, 1, 3, e, d - 1, d\}$. Then

$$A_{E,S}^{(d)} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ d & 0 & 0 & 0 & 1 & 0 \\ \binom{d}{3} & 0 & 0 & 0 & \binom{e}{2} & 0 \\ \binom{d}{e} & 0 & 0 & 1 & e & 1 \\ d & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

has determinant $(2e + 1)(e + 1)e/6 \neq 0$ and is hence invertible. So S is also not the support of an outcome in this case. \square

This finishes the proof of Theorem 7.2.

8. EXAMPLES AND DISCUSSION

In this paper, a theorem about the classification of discrete statistical models (Theorem 2.11) has motivated a combinatorial puzzle about chipsplitting games (Section 3): can the degree of a valid chipsplitting outcome grow indefinitely while the size of its support remains fixed? Theorem 3.6 answers this in the negative for certain support sizes. The theorem suggests a natural generalization.

Conjecture 8.1. *Let w be a valid outcome with a positive support of size $n + 1$. Then*

$$\deg(w) \leq 2n - 1.$$

In fact, we could have the right-hand side of the above inequality be any function of n and still be satisfied with the fact that the degree is bounded, as this would still guarantee a finite number of fundamental models in Δ_n . However, we know that the term $2n - 1$ is attained for infinitely many d .

Lemma 8.2. *Let $k \geq 0$ be an integer. Then*

$$t^{2k+1} + \sum_{i=0}^k \frac{2k+1}{2i+1} \binom{k+i}{2i} t^{k-i} (1-t)^{2i+1} = 1.$$

Proof. Let $S(k)$ denote the above sum and let $F(k, i)$ be its i -th summand. We find the recurrence

$$t^2 F(k-1, i) - (1-t)^2 F(k, i-1) - 2t F(k, i) + F(k+1, i) = 0$$

following Sister Celine’s method [6]. We sum over all integers i to obtain

$$t^2 S(k-1) - (1+t^2) S(k) + S(k+1) = 0.$$

Using this identity, it is easy to prove by induction on k that $S(k) = 1 - t^{2k+1}$, as required. □

Corollary 8.3. *Let $k \geq 0$ be an integer and let $w \in \mathbb{Z}^2$ be the chip configuration be defined by*

$$w_{0,0} = -1, \quad w_{2k+1,0} = 1, \quad w_{k-i,2i+1} = \frac{2k+1}{2i+1} \binom{k+i}{2i}$$

for $i \in \{0, 1, \dots, k\}$ and $w_{i,j} = 0$ otherwise. Then w is a valid outcome.

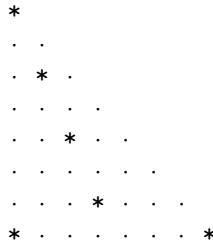


FIGURE 8. Support of the valid outcome defined in Corollary 8.3 for $k = 3$.

We conclude this paper with a discussion of computational results. Fixing a degree d , there are only finitely many fundamental outcomes of degree $\leq d$. It would be desirable to explicitly determine all of these and check that $d \leq 2n - 1$ holds for every computed outcome w with $\text{supp}^+(w) =: n + 1$. In principle one could check every possible subset $S \subseteq \{(i, j) \mid i + j \leq d\}$ for fundamental outcomes of support S , but this is computationally untractable. We were nevertheless able to carry out this computation for $d \leq 9$ and positive support size ≤ 6 using an optimization. The computer code for this is presented at <https://mathrepo.mis.mpg.de/ChipsplittingModels> along with a proof of its correctness. Table 1 shows an overview of our results. Thus, by the results of Sections 5–7, we now know that there are exactly 1, 4, 18, 134 fundamental models in $\Delta_1, \Delta_2, \Delta_3, \Delta_4$, respectively. We confirm that $d \leq 2n - 1$ holds for every computed outcome. We also notice that $n \leq d$ for all fundamental models we found. This is true in general, as the next proposition shows.

$n \setminus d$	1	2	3	4	5	6	7	8	9
1	1								
2		3	1						
3			12	4	2				
4				82	38	10	4		
5					602	254	88	24	2

TABLE 1. Number of fundamental outcomes of degree d with $\#\text{supp}^+(w) = n + 1$.

Proposition 8.4. *Let w be a degree- d fundamental outcome with $\#\text{supp}^+(w) = n + 1$. Then $n \leq d$.*

Proof. We recall that if \mathcal{M} is a fundamental model with support $S \subseteq \mathbb{Z}^2 \setminus \{(0, 0)\}$, then \mathcal{M} is the only model with support S . In terms of outcomes, this means that there exists a valid outcome w' with $\text{supp}^+(w') \subseteq S$ and that the space of outcomes whose support is contained in $S \cup \{(0, 0)\}$ is spanned by w' . In particular, this space must be 1-dimensional. When $n > d$, the space of chip configurations w' with $\text{supp}(w') \subseteq S \cup \{(0, 0)\}$ has dimension $> d + 1$. The subspace of outcomes has codimension $\leq d + 1$ and hence has dimension ≥ 2 in this case. So $n \leq d$. \square

Our computations show that for $n = 1, 2, 3, 4, 5$ there are 1, 1, 2, 4, 2 fundamental outcomes w with $\#\text{supp}^+(w) = n + 1$ and $\deg(w) = 2n - 1$, respectively. Taking into account that if w is a fundamental outcome then so is $(12) \cdot w$, most of these examples were already constructed in Proposition 8.3. The exceptions are the following two degree-7 fundamental outcomes.

2	1
. .	. .
. 7
.
. 7 . 7 .
.
. 7 . . . 7 7 . . .
-2 2	-1 1

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