3264 Questions about Symmetric Matrices

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What follows is a living document. It will evolve over time. It aims to inspire the study groups on *Linear Spaces of Symmetric Matrices* which are forming at MPI Leipzig in the summer of 2020. The coordinators for this are Orlando Marigliano, Mateusz Michałek, Tim Seynnaeve, and myself. Orlando, Mateusz and Tim are responsible for several of the questions below. I hope that you will speak to them. While everyone in the world is welcome to work on the suggested topics, we ask that you interact with some Leipzig-based people. If possible, please join one of the MPI teams. Doing so remotely is fine, of course.

We write $\mathrm{Sym}_2(K^n)$ for the $\binom{n+1}{2}$-dimensional vector space of symmetric $n \times n$-matrices with entries in a field K. These are identified with quadratic forms in $K[x_1, \ldots, x_n]$. Our favorite fields for geometry are the real numbers $K = \mathbb{R}$ and the complex numbers $K = \mathbb{C}$. A matrix in $\mathrm{Sym}_2(\mathbb{R}^n)$ is positive semidefinite if its quadratic form is nonnegative on $\mathbb{R}^n$. Such matrices form a closed convex cone, denoted PSD_n. The cone PSD_n is semialgebraic and full-dimensional, and it is self-dual under the trace inner product $(A, B) \mapsto \mathrm{trace}(AB)$.

A linear space of symmetric matrices (LSSM) is a linear or affine subspace $\mathcal{L} \subset \mathrm{Sym}_2(\mathbb{R}^n)$. We tacitly identify $\mathcal{L}$ with its image in the complex projective space $\mathbb{P}(\mathrm{Sym}_2(\mathbb{C}^n))$. We write $\mathcal{L}^\perp$ for the orthogonal complement under the trace inner product, and $\mathcal{L}^{-1}$ for the projective variety that is parametrized by the inverses of all invertible matrices in $\mathcal{L}$. The *spectrahedron* of $\mathcal{L}$ is the convex set $\mathcal{L} \cap \mathrm{PSD}_n$. This set is a *polyhedron* if $\mathcal{L}$ consists of diagonal matrices. The parameter space for all m-dimensional LSSM $\mathcal{L}$ is the Grassmannian $\mathrm{Gr}(m, \mathrm{Sym}_2(\mathbb{R}^n))$. We use notation from the textbook [Michałek-Sturmfels 2021], which we hope you have read.

Question 1. *What is a natural combinatorial notion of equivalence for LSSMs, which generalizes the concept of a matroid when $\mathcal{L} \subset \mathbb{R}^n$ consists of diagonal matrices. Same question for oriented matroids, positroids etc. This should relate to the combinatorics of spectrahedra.*

Question 2. *Characterize all subspaces $\mathcal{L}$ such that $\mathcal{L}^{-1}$ is also a linear space. The space of diagonal matrices is an obvious example, but there seem to be quite a few others.*

Question 3. *Characterize all subspaces $\mathcal{L}$ such that $\mathcal{L}^{-1}$ is a toric variety, possibly after a linear change of coordinates. This happens, for instance, for* Brownian motion tree models.

Question 4. *According to Hermann Schubert (1879) and his modern followers, the number of quadratic surfaces in $\mathbb{P}^3$ that are tangent to nine given quadratic surfaces is equal to* $666,841,088$. *Can the nine quadrics be chosen so that all* $666,841,088$ *solutions are real?*

Question 5. *The number of quadric hypersurfaces in $\mathbb{P}^{n-1}$ tangent to $\binom{n+1}{2} - 1$ given quadric hypersurfaces is given by the sequence 3264, 666841088, 48942189946470400, 641211464734373953791690014720, … What is the next number? What is the asymptotics?*

Question 6. *The* ML degree *of a variety in* $\mathrm{Sym}_2(\mathbb{C}^n)$ *is the number of critical points of the function* $M \mapsto \log(\det(M)) - \mathrm{trace}(S \cdot M)$ *for* S *generic. Characterize all linear spaces* $\mathcal{L}$ *of ML degree one. Same question for the reciprocal ML degree, i.e. the ML degree of* $\mathcal{L}^{-1}$.

Question 7. *What is the computational complexity of finding the ML degrees of an LSSM?*

Question 8. *The degree of* $\mathcal{L}^{-1}$ *is an upper bound for the ML degree of a LSSM* $\mathcal{L}$. *Often the two numbers are equal. What are necessary resp. sufficient conditions for this to happen?*

Question 9. *For a generic LSSM, how many of the complex critical points of the log-likelihood function in* $\mathcal{L}$ *resp. in* $\mathcal{L}^{-1}$ *are real, and how many of those are positive definite? Study the expected value for an appropriate probability distribution on* $\mathrm{Gr}(m, \mathrm{Sym}_2(\mathbb{R}^n))$.

Question 10. *If* $\mathcal{L}$ *is the space of Hankel matrices of size* n *then* $\mathcal{L}^{-1}$ *is the Grassmannian* $\mathrm{Gr}(2, n+1)$ *in its Plücker embedding. What is the generalization of this fact to catalecticants associated with forms in more than two variables? How about ternary quartics when* $n = 6$*?*

Question 11. *Determine the degree and ML degree for the Gaussian graphical model of the* m*-cycle. Explicit suggestions for these degrees appear in Conjecture 4.6 of [Sturmfels-Uhler].*

Question 12. *Interior point methods for semidefinite programming (SDP) travel along the central path. This is an algebraic curve inside the the spectrahedron* $\mathcal{L} \cap \mathrm{SDP}_n$. *Give a formula for its degree and genus, starting with the case when* $\mathcal{L}$ *is generic. The SFSU masters thesis of Joshua Rhodes, a student of Serkan Hoşten, contains concrete conjectures about this. In the special case of linear programming, this was resolved in [De Loera-Sturmfels-Vinzant].*

Question 13. *Generalize the notion of the* entropic *discriminant to the case of LSSM.*

Question 14. *Every LSSM* $\mathcal{L}$ *determines a linear projection of the PSD cone. Find a semialgebraic description for the image of that projection, in terms of invariants of* $\mathcal{L}$.

Question 15. *Every LSSM* $\mathcal{L}$ *of dimension* k *determines quadratic maps* $\mathbb{R}^n \to \mathbb{R}^k$ *and* $\mathbb{C}^n \to \mathbb{C}^k$. *Determine the subset of the Grassmannian for which each image fails to be closed.*

Question 16. *The* 3*-ellipses surrounding three given points in the plane are given by an LSSM* $\mathcal{L}$ *of dimension three in* $\mathrm{Sym}_2(\mathbb{R}^8)$. *Study the threefold* $\mathcal{L}^{-1}$. *Determine its degree, equations and singularities. A related question: does every Riemann surface of degree three arise from some* 3*-ellipse? Yuhan Jiang has some partial results on their canonical models.*

Question 17. *Classify combinatorial types of quintic spectrahedra in* $\mathbb{R}^3$. *Can you realize all* 20 *complex singular points of the quintic symmetroid in the boundary of its spectrahedron?*

Question 18. *Show that the reciprocal ML degree of a generic linear subspace* $\mathcal{L}$ *in* PSD_n *is a polynomial in* n *of degree* $\dim(\mathcal{L})$. *This is Conjecture 4.2 in [Sturmfels-Timme-Zwiernik].*

Question 19. *Find a formula for the reciprocal ML degree of an arbitrary linear space of diagonal matrices. Is this ML degree a matroid invariant?*

Question 20. *Find a formula for the* Hurwitz form *of symmetric* 4×4 *matrices of rank* 2. *This is a polynomial of degree* 30 *in Plücker coordinates. This would be applicable to the essential variety in computer vision. Recall the nice Pfaffian formula for the* Chow form *due to [Floystad-Kileel-Ottaviani], which rests on Ulrich sheaves and representation theory.*

Question 21. *Two-parameter eigenvalue problems link two LSSM* $\{A_i+\lambda B_i+\mu C_i\}$, $i = 1, 2$. *What is the algebro-geometric meaning of the hypothesis in Henrik's NASO talk on April 30?*

Question 22. *What can be said about the image of the PSD cone in the space of complete quadrics? This should be an interesting semialgebraic set, related to SDP complementarity.*

Question 23. *For* $n = 3$ *and* $n = 4$*, find generators for the multihomogeneous prime ideal of the natural embedding of the space of complete quadrics into a product of projective spaces.*

Question 24. *Which matrix ranks occur in the boundary of* Gram spectrahedra*?*

Question 25. *Let* $\mathcal{L}$ *be the* $\binom{g-2}{2}$*-dimensional space of quadrics defining a canonical curve in* $\mathbb{P}^{g-1}$*. What is the variety* $\mathcal{L}^{-1}$*? How does it vary over the moduli space* $\mathcal{M}_g$*?*

Question 26. *Same question as the previous one, but for Grassmannians, Segre varieties, Veronese varieties, and your other favorite projective varieties that are defined by quadrics.*

Question 27. *What can we say about the likelihood geometry of the space of symmetric tridiagonal matrices? Same question for pentadiagonal matrices and similar banded patterns.*

Question 28. *Consider the variety in* $\mathbb{P}(\mathrm{Sym}_2(K^n))^2$ *that is parametrized by* (X, X^{-1})*. Show that its prime ideal is generated by the obvious bilinear equations, namely the off-diagonal entries of the matrix product* XY *plus differences of diagonal entries. What about syzygies? Can we find a nice Gröbner basis for this ideal? Are the matrix entries a Khovanskii basis?*

Question 29. *For any LSSM* $\mathcal{L}$*, consider its image in the previous variety, or in the space of complete quadrics. What is its multidegree, especially for* $\mathcal{L}$ *that arise in statistics?*

Question 30. *The multidegree of the previous variety is a generating function* $\sum_d \phi(d,n)z^d$*. Tim, Mateusz and collaborators proved Bernd & Caroline's conjecture that* $\phi(d,n)$ *is a polynomial in* n *of degree* $d-1$*. Do the coefficients of this polynomial form a log-concave sequence?*

Question 31. *Can we generalize the polynomiality result above? To be precise, let us fix numbers* $d_0, d_1, \ldots d_k$*. For* n *large enough, let* $p(n)$ *be the number of quadric hypersurfaces in* $\mathbb{P}^{n-1}$ *passing through* d_0 *given points, tangent to* d_1 *given lines, ..., tangent to* d_k *given* k*-planes, and tangent to* $\binom{n+1}{2} - 1 - \sum d$ *given hyperplanes. Is* $p(n)$ *a polynomial in* n*?*

Question 32. *In nonarchimedean analysis, e.g. in the study of Gaussians by [El Maazouz-Tran], it is natural to replace the PSD cone by the affine building of* $\mathrm{SL}_2(K)$*, where* K *is a local field. What can be said about LSSM and complete quadrics in that setting of buildings?*

Question 33. *Prove Conjecture 6.3 in [Boege-Kahle-Sturmfels], i.e. the ideal of homogeneous relations among principal and almost-principal minors of a symmetric matrix is generated by quadrics. Does the view on complete quadrics add any information about Gaussoids?*

Question 34. *In 1977-1980, CTC Wall published three papers on nets of quadrics. From today's perspective, how to think about his results? What about geometric invariant theory?*

Question 35. *A point $\mathcal{L}$ in $\mathrm{Gr}(m, \mathrm{Sym}_2(\mathbb{C}^n))$ gives a hypersurface of degree n in $\mathbb{P}^{m-1}$ and a subvariety of $\mathbb{P}^{n-1}$ defined by m quadrics. How are these related? For instance, for $m = 3$, this links plane quartics and Cayley octads ($n = 4$) and plane sextics and K3 surfaces ($n = 6$).*

Question 36. *Antonio Lerario's early work used the duality above to study the topology of quadratically defined semialgebraic sets. How to turn his methods into practical algorithms?*

Question 37. *For $1 \leq i < j \leq n$ let $M_{i,j}$ be the symmetric $\binom{n}{2} \times \binom{n}{2}$ matrix with rows and columns indexed by pairs in $\{1, \ldots, n\}$. The $(\{a, b\}, \{c, d\})$ entry of $M_{i,j}$ equals 1 if $a = i, b = c, d = j$, or $c = i, d = a, b = j$, and 0 otherwise. Let $\mathcal{L}$ be the space spanned by all $\binom{n}{2}$ matrices $M_{i,j}$. How many, as a function of n, rank one matrices are needed to span a space that contains $\mathcal{L}$? What is the minimum α such that n^α suffices for n large enough?*

Question 38. *Suppose we have a suitable notion of matroid for LSSMs. What are some nonrealizable matroids? Do they encode incidence theorems? What are smallest examples?*

Question 39. *Figure 5 in [Wall 1977] shows the stratification of $\mathrm{Gr}(3, \mathrm{Sym}_2(\mathbb{C}^3))$ according to LSSM types. How does it compare to the usual matroid stratification of $\mathrm{Gr}(3, 6)$ for rank 3 matroids on 6 elements? Is Wall's stratification the one we want for nonabelian matroids?*

Question 40. *If f is a cubic form then its Hessian $\mathcal{H}_f$ is an LSSM. What can we say about $\mathcal{H}_f$ from the perspectives of SDP and statistics? The* Sylvester spectrahedron *is a first nice example. Express invariants such as $\mathrm{degree}(\mathcal{H}_f^{-1})$ and the two ML degrees in terms of f.*

Question 41. *For a real affine LSSM $\mathcal{L}$ with empty spectrahedron, the subset of all convex quadrics in $\mathcal{L}$ is a convex subset of $\mathcal{L}$. Following [Gouveia-Parrilo-Thomas], this convex set represents the first theta body of the variety defined by $\mathcal{L}$. Study these objects for $n = 4$. Use this to identify convex hulls of canonical curves of genus 5 and quartic del Pezzo surfaces.*

Question 42. *The last section of [Sturmfels-Uhler] contains a census of all LSSM that represent colored Gaussian graphical models on the 4-cycle. Can we identify some general patterns from these data? What does the discrepency between degree and ML-degree tell us?*

Question 43. *Can we define a generalization of the space of complete quadrics where the role of the symmetric determinant is played by an arbitrary hyperbolic polynomial f? Such a manifold could be a canonical resolution of the graph of the gradient map of f.*

Question 44. *For a homogeneous real polynomial f of degree $d \geq 3$, we construct an LSSM $\mathcal{L}_f$ by taking the space of all derivatives of f of order $d-2$. What can we say about $\mathcal{L}_f$ and its geometry? The Brändén-Huh theory of Lorentzian polynomials might offer some clues.*

Question 45. *Ellingsrud-Piene-Stromme [1985] studied the subvariety of $\mathrm{Gr}(3, \mathrm{Sym}_2(\mathbb{C}^4))$ whose points are nets of quadrics defining twisted cubic curves in $\mathbb{P}^3$. Can you find generators for the prime ideal of that variety? Same question for rational normal curves in $\mathbb{P}^{n-1}$.*

Question 46. *Complete intersections of two quadrics have very interesting Fano schemes. It includes Jacobians of genus* 2 *curves (d'aprés André Weil) and blow-ups of* $\mathbb{P}^n$ *at* $2n+3$ *points (by [Araujo-Casagrande]). What does this mean for computations, statistics and convexity?*

Question 47. *Consoder the space of symmetric* $n \times n$ *matrices whose off-diagonal entries are all equal. Following Proposition 4.6 of [Sturmfels-Uhler-Zwiernik], we conjecture that the ML degree is* $2^n - n - 1$ *and the reciprocal ML degree is* $2^{n+1} - 2n - 3$*. Is this true or false?*

Question 48. *Fix two graphs* G *and* H *on* $\{1, \ldots, n\}$*. These specify supports for a reciprocal pair of matrices* (X, X^{-1}) *in* $(\mathrm{PSD}_n)^2$*. Study the geometry of this model. Is it smooth?*

Question 49. *What are the secant varieties of complete intersections of quadrics? Can they be degenerate? Focus on the case of secant hypersurfaces. For instance, an LSSM in* $\mathrm{Gr}(3, \mathrm{Sym}_2(\mathbb{C}^5))$ *defines a genus* 5 *canonical curve. Its secant variety is a hypersurface of degree* 16 *in* $\mathbb{P}^4$*. How to write its equation in terms of the three symmetric* 5×5 *matrices?*

Question 50. *How can an LSSM intersect the various varieties of partitioned eigenvalues? An example is the* 5*-dimensional projective variety of real symmetric* 4×4 *matrices with two double eigenvalues. Study the Chow form, Hurwitz form and discriminant for this variety.*

Question 51. *The natural CAT(0) metric on* PSD_n *represents the Wasserstein distance between Gaussians. How does the geometry of its geodesic segments and geodesic polytopes compare to that of LSSMs? How to compute the CAT(0) distance between two LSSMs?*

Question 52. *Given three matrices in* PSD_n*, we can consider their geodesic triangle for the metric in the previous question. This is the smallest superset of the three matrices which is closed under taking geodesics. What is the geometry of such triangles? Can they have arbitrarily high dimensions? This holds for the BHV metric on the space of phylogenetic trees.*

Question 53. *A matrix in* PSD_n *is an* M-matrix *if its off-diagonal entries are nonpositive. Consider the problem of maximizing the log-likelihood function, for any* S*, over the convex cone of positive definite M-matrices. What subsets of this cone arises as optimal sets?*

Question 54. *Which groups arise as stabilizers for the natural congruence action of* $\mathrm{GL}(n)$ *on* $\mathrm{Gr}(m, \mathrm{Sym}_2(\mathbb{R}^n))$*? Describe this stabilizer group explicitly when* $\mathcal{L}$ *is a graphical model.*

Question 55. *Fix* m *and* n*. The* Pataki range *is the set of integers* r *such that, for a generic LSSM* $\mathcal{L}$*, the projective dual of the determinantal variety* $\mathcal{L} \cap \{\mathrm{rank} \leq r\}$ *is a hypersurface, The rank of every extreme ray of the spectrahedron* $\mathcal{L} \cap \mathrm{SDP}_n$ *is in that range. Can we find one single* $\mathcal{L}$ *which simultaneously realizes all ranks in the Pataki range by its extreme rays?*

Question 56. *What can real algebraic geometry say about the Burer-Monteiro method, notably in the setting studied by [Cifuentes-Moitra] ? Why are there no spurious local optima?*

Question 57. *How to construct instances of high-dimensional LSSM that do not contain low rank matrices? What about LSSM that have low rank matrices over* $\mathbb{C}$ *but not over* $\mathbb{R}$*?*

Question 58. *Cryptographers are interested in LSSM over finite fields. Why? How are their results and problems related to what we know over* $\mathbb{R}$ *and* $\mathbb{C}$*? Who is John Sheekey?*

Question 59. *Is there a holonomic D-module we can naturally associate with an LSSM? Here "naturally" should also mean "statistically meaningful" for the case of graphical models.*

Question 60. *Fix integers* m, n, t *and consider the semialgebraic subset of* $\mathrm{Gr}(m, \mathrm{Sym}_2(\mathbb{R}^n))$ *whose points are the LSSM of maximum likelihood threshold* t*. What is its dimension?*

Question 61. *Study the spectrahedral cones defined by Brownian motion tree models. Which matrix ranks occur among the extreme rays of these cones? How does it depend on the tree?*

Question 62. *Which rational projective varieties have the form* $\mathcal{L}^{-1}$ *for some LSSM* $\mathcal{L}$*?*

Question 63. *Every finite-dimensional* $\mathbb{C}$*-algebra* A *gives rise to a bilinear map* $A \times A \to A$*, via the multiplication map. Assuming* A *to be commutative, this determines an LSSM. How are structural properties of the algebra* A *related to the geometry of the associated LSSM?*

Question 64. *Two general symmetric* $n \times n$ *matrices can be simultaneously diagonalized. This implies that two general quadrics in* n *variables have a simultaneous Waring decomposition of rank* n*. What is the simultaneous Waring rank for three or more general quadrics?*

Question 65. *On July 6, at 16:17h, Rosa Winter asked about the discriminant in* $\mathrm{Gr}(m, \mathrm{Sym}_2(\mathbb{C}^n))$ *of all non-generic subspaces* $\mathcal{L}$*. Mateusz responded that the non-genericity condition is* $\mathcal{L}^{\perp} \cap \mathcal{L}^{-1} \neq \emptyset$*. Isn't this (more or less) the same as the condition for* $\mathcal{L}$ *to be in the bad hypersurfaces of [Jiang-Sturmfels], defined by* $\mathcal{L} \times \mathcal{L}^{\perp}$ *meeting conormal varieties?*

... to be followed by **3199** additional questions.